A Decidable Variety that is Finitely Undecidable

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Abstract

We construct a decidable first-order theory T such that the theory of its finite models is undecidable. Moreover, T will be equationally axiomatizable and of finite type.

1 Introduction

A class $\mathcal{K}$ of first-order structures in the same language is said to be *decidable* if the theory of $\mathcal{K}$ is decidable (i.e., recursive). $\mathcal{K}$ is *finitely decidable* if the class $\mathcal{K}_{\text{fin}}$ of finite members of $\mathcal{K}$ is decidable. A theory T is *finitely decidable* if the theory T_{fin} of the class of finite models of T is decidable.

There is some positive correlation between the decidability and the finite decidability. For instance, for locally finite strongly abelian varieties of finite type, those two concepts coincide [MV89]. Although there is some tendency that finite decidability implies decidability, it is not hard to find counterexamples in the literature to this implication—two such examples are the theory of the variety generated by the four element ring $\mathbb{Z}_4$ and the theory of monadic algebras, which appear in [Zam78] and [Com69] respectively. On the other hand, there has been no example of a decidable first-order theory which is finitely undecidable. Later, in section 6 of this article, we will see some situations when the implication "decidable $\Rightarrow$ finitely decidable" is guaranteed to hold.

The chief method of proving that a theory T is finitely undecidable is to interpret (i.e., semantically embed) a class of hereditarily finitely undecidable class, such as the class of all finite graphs, into the class of finite models of T [ELTT65]. But then T is not only finitely undecidable but also hereditarily undecidable. Hence the theory T itself, being a subtheory of the theory T_{fin} of the finite models of T, becomes undecidable. So we see that the method of semantic embedding cannot be used in finding a theory which is decidable but not finitely so. Nevertheless, in case we impose no extra condition on the theory T, it is not too difficult to produce such an example, which is given below.

1.1 Example We construct a decidable theory T such that the theory T_{fin} of its finite models is undecidable. The language L of T consists of a single unary predicate symbol

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$\mathtt{P}$. Given an L-structure, let P denote the interpretation of $\mathtt{P}$ in the structure, and let P' denote the set theoretic complement of P in the universe of the structure. Choose any recursive function $h : \omega \to \omega$ with a nonrecursive range $S \subseteq \omega$. For each $n \in \omega$, let σ_n be a sentence in our language L saying that

$$\big(|P| = n\big) \Rightarrow \big(|P'| = h(n)\big).$$

Let our theory T have $\{\sigma_n \mid n \in \omega\}$ as its set of axioms. For each $m \in \omega$, let τ_m be an L-sentence saying that $|P'| = m$. Then for each $m \in \omega$, we have the following biimplication:

$$m \notin S \Leftrightarrow \neg\tau_m \in T_{\mathrm{fin}}.$$

This is nothing but the contrapositive of the fact that

$$m \in S \Leftrightarrow \tau_m \text{ holds in some finite model of } T.$$

Thus the decidability of T_{fin} would imply the recursiveness of S, which contradicts our choice of h. We have shown that T_{fin} is undecidable.

Now to show that T is decidable, we will show that there exists an algorithm to decide whether a given L-sentence is consistent with T. Let σ be an arbitrary L-sentence. We want to know whether $T \cup \{\sigma\}$ has a model. First we claim that we can effectively decide whether σ has a model in which P is infinite. To prove this, we use the fact that the predicate calculus in the language $\{\mathtt{P}, \mathtt{f}\}$, where $\mathtt{f}$ is a unary function symbol, is decidable[1] [Ehr59]. It is clear that σ has a model in which P is infinite iff the following sentence is consistent:

$$\sigma \wedge \big(f \text{ is 1–1 on } P \text{ and } f(P) \text{ is a proper subset of } P\big).$$

This proves our claim.

Now given σ, we first determine whether σ has a model in which P is infinite. If the test result is positive, then obviously $T \cup \{\sigma\}$ has a model. In the other case, the cardinality of P in the models of σ is bounded by a finite cardinal, say N. (This is easily proved by the compactness theorem.) Moreover, we can effectively find such an N by recursively checking the logical validity of the following sentences:

$$\sigma \Rightarrow |P| \leq k, \quad (k = 0, 1, \dots).$$

We may take N to be the first $k < \omega$ for which the above sentence is logically valid. Now see if

$$\sigma \wedge \big(|P| = k\big) \wedge \big(|P'| = h(k)\big)$$

is consistent for $k = 0, 1, \dots, N$. In case at least one of those sentences has a model, say $\mathbf{A}$, then $T \cup \{\sigma\}$ has a model, namely $\mathbf{A}$. Otherwise $T \cup \{\sigma\}$ has no model. $\square$

[1] This fact is first announced in [Ehr59]. An extended version is found in [Rab69].

The main goal of this paper is to construct an *equationally axiomatizable* theory T in a language of algebras (i.e., a theory of a variety) such that T is decidable while T_{fin} is not. In the example above it was essential that there exist first-order formulas that *talk about (finite) cardinalities* of definable subsets—which would be very hard for *equations* if not impossible. So in constructing our theory T, we specify the generators of its model class $\mathcal{V} \stackrel{\text{def}}{=} \{\text{all models of } T\}$ in some clever way instead of explicitly describing the axioms of T. The class $\mathcal{W}$ of generators of $\mathcal{V}$ will be a model class of a universal theory: i.e., a theory which has a set of axioms consisting of universal sentences.

We assume that the reader has a nodding knowledge in mathematical logic and model theory. Chang-Keisler [CK90] would be an excellent reference. For general universal algebraic terminology the reader is referred to [MMT87]. For the theory of discriminators and Boolean products the reader is referred to [BS81, BW79], and for the Feferman-Vaught type theorem to [BW79, FV59].

2 A Universally Axiomatizable Example

Let a first-order language L^1 consist of four symbols $\mathtt{f}$, $\mathtt{g}$, $\mathtt{0}$ and $\mathtt{1}$, where $\mathtt{f}$, $\mathtt{g}$ are unary function symbols and $\mathtt{0}$, $\mathtt{1}$ are constant symbols. For $q \geq 0$, we use $\mathtt{f}^q(x)$ as an abbreviation of the term $\mathtt{f} \cdots \mathtt{f}x$ in which $\mathtt{f}$ occurs exactly q times.

Throughout we adopt the following conventions: If Φ is a finite set of L^1-formulas, then $\bigwedge \Phi$ (resp. $\bigvee \Phi$) is uniquely determined: i.e., the order of the conjuncts (resp. disjuncts) is determined by some fixed enumeration of the set of all L^1-formulas. The conjunction (resp. disjunction) of the empty set of formulas is $\mathtt{0} \approx \mathtt{0}$ (resp. $\mathtt{0} \not\approx \mathtt{0}$), where $\approx$ is the formal equality symbol. A finite sequence $\langle a_1, \ldots, a_n\rangle$ is usually abbreviated to $\bar{a}$. If φ is a first-order formula and $\bar{x} = \langle x_1, \ldots, x_n\rangle$ is a sequence of n-distinct variables, then a model for a formula $\varphi(\bar{x})$ means a structure $\mathbf{A}$ together with an n-tuple $\bar{a} \in A^n$. That is, $x_i^{\langle \mathbf{A}, \bar{a}\rangle} = a_i$ for each $i = 1, \ldots, n$. If $a, b \in \prod_{i \in I} A_i$, then the *equalizer* $[\![a = b]\!]$ is the subset of I defined by $[\![a = b]\!] \stackrel{\text{def}}{=} \{i \in I \mid a(i) = b(i)\}$. In general if R_i is an n-ary relation on A_i, then $[\![R_i(\bar{a})]\!] \subseteq I$ is defined to be $\{i \in I \mid A_i \models R_i(a_1(i), \ldots, a_n(i))\}$. If A is a set then $|A|$ denotes the cardinality of A. For set theoretic difference we use the symbol '$-$': i.e., $A - B \stackrel{\text{def}}{=} \{x \in A \mid x \notin B\}$.

2.1 Definition Fix a recursive function $h : \omega \to \omega$ with a nonrecursive range. Let Σ^1 be the set of following sentences in L^1, and let $\mathcal{K}^1$ be the class of all models of Σ^1. Let T^1 be the theory of $\mathcal{K}^1$. Note that each member of Σ^1 is universal.

(a) $\forall x(\mathtt{f}(\mathtt{g}(x)) \approx x \approx \mathtt{g}(\mathtt{f}(x)))$,

(b) $\mathtt{0} \not\approx \mathtt{1}$,

(c) $\forall x \forall y\big((\mathtt{f}^n(x) \approx x) \to \bigvee_{i<n}(\mathtt{f}^i(x) \approx y)\big), \quad n \geq 1$,

(d) $\big((\mathtt{f}^n(\mathtt{0}) \approx \mathtt{1}) \wedge \bigwedge_{i<n}(\mathtt{f}^i(\mathtt{0}) \not\approx \mathtt{1})\big) \to \big((\mathtt{f}^{h(n)}(\mathtt{1}) \approx \mathtt{0}) \wedge \bigwedge_{i<h(n)}(\mathtt{f}^i(\mathtt{1}) \not\approx \mathtt{0})\big), \quad n \geq 1.$

Let Σ^0 be the subset of Σ^1 consisting of sentences of the type $(a), (b)$ and (c) above, and let $\mathcal{K}^0$ be the class of all models of Σ^0. Let T^0 be the theory of $\mathcal{K}^0$. ◀

In order to describe our models conveniently we need the following definition.

2.2 DEFINITION

(a) In an L^1-structure $\mathbf{A}$, a *cycle* is defined to be a subset C of the universe A such that

$$(\forall x, y \in C)\big(f(x) \in C \text{ and } (\exists n < \omega)(f^n(x) = y \text{ or } f^n(y) = x)\big).$$

(b) For each $m > 1$, define L^1-sentences as follows:

$$\begin{aligned}
\mathtt{Cycle}_m &= (\mathtt{f}^m(\mathtt{0}) \approx \mathtt{0}) \wedge \bigwedge_{0<i<m} (\mathtt{f}^i(\mathtt{0}) \not\approx \mathtt{0}), \\
\mathtt{Cycle}_{\leq m} &= \bigvee_{1<i\leq m} \mathtt{Cycle}_m, \\
\mathtt{Cycle}_{>m} &= \neg\mathtt{Cycle}_{\leq m}.
\end{aligned}$$

(c) For each $n \in \omega$ and for each L^1-term t and s,

$$t + n \approx s \text{ is an abbreviation for } (\mathtt{f}^n(t) \approx s) \wedge \bigwedge_{i<n} (\mathtt{f}^n(t) \not\approx s).$$

◀

Note that the members of $\mathcal{K}^1$ are exactly the L^1-structures in which

(a) f and g are both bijections and inverses to each other,

(b) $0 \neq 1$,

(c) if a finite cycle exists, then it is the whole universe of the structure, otherwise, the universe is the disjoint union of the denumerable cycles,

(d) for $n > 0$, if $0 + n = 1$, then 0 and 1 lie in the unique finite cycle and $1 + h(n) = 0$.

If we remove condition (d) from above, then we get the class $\mathcal{K}^0 \supseteq \mathcal{K}^1$.

We are going to show that $\mathcal{K}^1$ is decidable but not finitely decidable. The proof of the undecidability of $\mathcal{K}^1_{\text{fin}}$ is easy—it is more or less the same as the undecidability proof in example 1.1 in the previous section: i.e., for each $m > 0$, let τ_m be an L^1-sentence saying that $1 + m = 0$. Then we have, for all $m > 0$,

$$(m \in h(\omega^+) - \{0\}) \Leftrightarrow (\tau_m \text{ has a model in } \mathcal{K}^1_{\text{fin}}),$$

where ω^+ is the set of positive integers. Therefore $\mathcal{K}^1_{\text{fin}}$ is undecidable.

The proof of the decidability of $\mathcal{K}^1$ is harder—it is given at theorem 2.17 after a series of lemmas, which we begin by proving some facts on the "base theory" T^0.

2.3 Lemma *The theory T^0 is decidable.*

Proof. Clearly T^0 is recursively enumerable. Thus it suffices to show that the complement of T^0 is recursively enumerable. We will do this by proving the following claim.

Claim: for each L^1-sentence σ,

$$\begin{aligned}\sigma \notin T^0 \Leftrightarrow\ & (\exists m > 1)(\exists q < m)(\Sigma^2_{m,q} \vdash \neg\sigma) \text{ or} \\ & (\exists q > 0)(\Sigma^3_q \vdash \neg\sigma) \text{ or} \\ & (\exists q > 0)(\Sigma^4_q \vdash \neg\sigma) \text{ or} \\ & \Sigma^5 \vdash \neg\sigma, \end{aligned} \tag{1}$$

where

$$\Sigma^2_{m,q} \stackrel{\text{def}}{=} \Sigma^0 \cup \{\mathtt{Cycle}_m, \mathtt{0} + q \approx \mathtt{1}\}, \quad (m > 1,\ 0 < q < m) \tag{2}$$

$$\Sigma^3_q \stackrel{\text{def}}{=} \Sigma^0 \cup \{\mathtt{Cycle}_{>m} \mid m > 1\} \cup \{\mathtt{0} + q \approx \mathtt{1}\}, \quad (q > 0) \tag{3}$$

$$\Sigma^4_q \stackrel{\text{def}}{=} \Sigma^0 \cup \{\mathtt{Cycle}_{>m} \mid m > 1\} \cup \{\mathtt{1} + q \approx \mathtt{0}\}, \quad (q > 0) \tag{4}$$

$$\Sigma^5 \stackrel{\text{def}}{=} \Sigma^0 \cup \{\mathtt{Cycle}_{>m} \mid m > 1\} \cup \{\mathtt{0} + q \not\approx \mathtt{1}, \mathtt{1} + q \not\approx \mathtt{0} \mid q > 0\}. \tag{5}$$

Note that (2) is complete (for each m and q) because it has only one model up to isomorphism. (3), (4) and (5) are all complete because each of them has no finite model and categorical in any uncountable cardinality (the Łoś-Vaught test).

Now to prove $\Rightarrow$ of (1), suppose $\sigma \notin T^0$. Then $T^0 \cup \neg\sigma$ has a model which must be a model of one among (2) – (5), say $\Sigma^2_{m,q}$ for some $m > 1$ and $0 < q < m$. Then $\Sigma^2_{m,q} \cup \{\neg\sigma\}$ is consistent. But $\Sigma^2_{m,q}$ is complete. Thus $\Sigma^2_{m,q} \vdash \neg\sigma$ follows. The other cases (3), (4) and (5) are handled similarly. This proves $\Rightarrow$.

The other direction $\Leftarrow$ is obvious. □

2.4 Definition

(*a*) Given $m \geq 2$, we define the following subclasses of $\mathcal{K}^0$:

$$\begin{aligned} \mathcal{K}^0_m &\stackrel{\text{def}}{=} \{\mathbf{A} \in \mathcal{K}^0 \mid \mathbf{A} \models \mathtt{Cycle}_m\}, \\ \mathcal{K}^0_{\leq m} &\stackrel{\text{def}}{=} \{\mathbf{A} \in \mathcal{K}^0 \mid \mathbf{A} \models \mathtt{Cycle}_{\leq m}\}, \\ \mathcal{K}^0_{>m} &\stackrel{\text{def}}{=} \mathcal{K}^0 - \mathcal{K}^0_{\leq m}, \\ \mathcal{K}^0_{\infty} &\stackrel{\text{def}}{=} \{\mathbf{A} \in \mathcal{K}^0 \mid \mathbf{A} \models \mathtt{Cycle}_{>m} \text{ for all } m \geq 2\}. \end{aligned}$$

We define $\mathcal{K}^1_m$, $\mathcal{K}^1_{\leq m}$, $\mathcal{K}^1_{>m}$ and $\mathcal{K}^1_\infty$ in the obvious way.

(*b*) Let a sequence $\bar{x} \stackrel{\text{def}}{=} \langle x_1, \dots, x_n\rangle$, $(n \geq 0)$ of pairwise distinct variables be given. For each $m \geq 2$ and $k \geq 0$, we define a set of formulas

$$\Theta^k_m(\bar{x}) \stackrel{\text{def}}{=} \{\, \theta(\bar{x}) \mid \theta(\bar{x}) = \bigwedge \Psi^k_{\mathbf{A},\bar{a}},\ \mathbf{A} \in \mathcal{K}^0_m,\ \bar{a} \in A^n\}, \tag{6}$$

where for each $k \geq 0$, $\mathbf{A} \in \mathcal{K}^0$ and $\bar{a} \in A^n$,

$$\Psi^k_{\mathbf{A},\bar{a}} \stackrel{\text{def}}{=} \{\, \psi(\bar{x}) \mid \mathbf{A} \models \psi(\bar{a}),\ \psi = \mathtt{f}^q(x) \approx y \text{ or } \psi = \mathtt{f}^q(x) \not\approx y \\ \text{for some } 0 \leq q \leq k \text{ and } \{x, y\} \subseteq \{x_1, \dots, x_n, \mathtt{0}, \mathtt{1}\}\}. \quad (7)$$

In (6), recall that the conjunction $\theta(\bar{x})$ is uniquely determined (in syntactical sense) from the set $\Psi^k_{\mathbf{A},\bar{a}}$ by a remark in the second paragraph of this section. We let

$$\Theta_m(\bar{x}) \stackrel{\text{def}}{=} \Theta^{m-1}_m(\bar{x}) \text{ for } m < \omega,$$

and let

$$\Theta^k_\infty(\bar{x}) \stackrel{\text{def}}{=} \{\, \theta(\bar{x}) \mid \theta(\bar{x}) = \bigwedge \Psi^k_{\mathbf{A},\bar{a}},\ \mathbf{A} \in \mathcal{K}^0_\infty,\ \bar{a} \in A^n\}.$$

◀

The idea behind this definition is as follows. $\Theta_m(\bar{x})$ is intended to be a representative set of all quantifier free n-types of $\mathcal{K}^0_m$. Θ^k_∞ is intended to be a representative set of all quantifier free n-types of $\mathcal{K}^0_\infty$ where the "level of description" is bounded by k to ensure finiteness. The point is that for each formula φ, we can recursively find a large enough bound k_φ that "works", and further, the resulting "bounded" n-type works not only for $\mathcal{K}^0_\infty$ but also for $\mathcal{K}^0_{>M}$ for sufficiently large $M \in \omega$ which is obtained recursively.

We rigorously develop this idea in the following lemmas. From now on, to the end of this paper we use $\bar{x}$ as an abbreviation of the sequence $\langle x_1, \dots, x_n \rangle$ of n distinct variables for some fixed $n \geq 0$. First we state the following easily proved lemma.

2.5 Lemma *Given $m \geq 2$ and $\Theta_m(\bar{x})$,*

(a) $\Theta_m(\bar{x})$ is finite,

(b) Each $\theta \in \Theta_m(\bar{x})$ has a model in $\mathcal{K}^0_m$,

(c) $\mathcal{K}^0_m \models \bigvee \Theta_m(\bar{x})$,

(d) $\mathcal{K}^0_m \models \neg(\theta \wedge \theta')$ for all $\theta \neq \theta' \in \Theta_m(\bar{x})$.

(e) For $\theta \in \Theta_m(\bar{x})$, $\mathbf{A} \in \mathcal{K}^0_m$ and $\bar{a} \in A^n$, $\mathbf{A} \models \theta(\bar{a})$ iff $\theta = \bigwedge \Psi^{m-1}_{\mathbf{A},\bar{a}}$.

Proof. (a), (b) and (c) are trivial. To prove (d), just note that for every $0 \leq q \leq m$ and for every $\{x, y\} \subseteq \{x_1, \dots, x_n, \mathtt{0}, \mathtt{1}\}$, either $\mathtt{f}^q(x) \approx y$ or $\mathtt{f}^q(x) \not\approx y$ appears as a conjunct of θ and the same is true for θ'. So $\theta \neq \theta'$ means that for some q, x and y, $\mathtt{f}^q(x) \approx y$ is a conjunct of only one of $\{\theta, \theta'\}$ and $\mathtt{f}^q(x) \not\approx y$ is a conjunct of the other. This, in fact, shows that $\models \neg(\theta \wedge \theta')$. (e) follows from (d) easily. □

2.6 Definition Given an L^1-formula $\varphi(\bar{x})$ and $m \geq 2$, we let

$$\Theta_m(\varphi) \stackrel{\text{def}}{=} \{\theta \in \Theta_m(\bar{x}) \mid \mathcal{K}^0_m \models \theta \to \varphi\}, \text{ and let}$$

$$\theta_m(\varphi) \stackrel{\text{def}}{=} \bigvee \Theta_m(\varphi).$$

◀

Note that $\mathcal{K}_m^0$ is decidable because its theory is a finite extension of the theory T^0 which was proved to be decidable in lemma 2.3. Thus, from lemma 2.5.(a), the process $\langle \varphi, m\rangle \mapsto \theta_m(\varphi)$ is effective.

2.7 Lemma *Given any L^1-formula $\varphi(\bar{x})$ and $m \geq 2$, we have*

$$\mathcal{K}_m^0 \models \varphi \leftrightarrow \theta_m(\varphi).$$

Proof. $\leftarrow$ is obvious from the construction of $\theta_m(\varphi)$.

To prove $\rightarrow$, suppose that $\mathbf{A} \in \mathcal{K}_m^0$ and $\bar{a} \in A^n$ are given so that $\mathbf{A} \models \varphi(\bar{a})$. We want to show $\mathbf{A} \models \theta_m(\varphi)(\bar{a})$.

Let $\theta(\bar{x}) = \bigwedge \Psi_{\mathbf{A},\bar{a}}^{m-1}$ where $\Psi_{\mathbf{A},\bar{a}}^{m-1}$ is as defined in (7). Obviously $\mathbf{A} \models \theta(\bar{a})$. In order to show $\mathbf{A} \models \theta_m(\varphi)(\bar{a})$, it suffices to show that $\theta(\bar{x}) \in \Theta_m(\varphi)$, or equivalently, $\mathcal{K}_m^0 \models \theta(\bar{x}) \rightarrow \varphi(\bar{x})$.

Suppose that $\mathbf{B} \in \mathcal{K}_m^0$ and $\bar{b} \in B^n$ satisfy $\mathbf{B} \models \theta(\bar{b})$. We want to show that $\mathbf{B} \models \varphi(\bar{b})$. Define a map $\alpha : A \rightarrow B$ by

$$\alpha((\mathtt{f}^{\mathbf{A}})^{\ell}(\mathtt{0}^{\mathbf{A}})) \stackrel{\text{def}}{=} (\mathtt{f}^{\mathbf{B}})^{\ell}(\mathtt{0}^{\mathbf{B}}) \quad \text{for } 0 \leq \ell < m. \tag{8}$$

Then we claim that α is an isomorphism from $\mathbf{A}$ onto $\mathbf{B}$ and moreover $\alpha(a_i) = b_i$ for all $1 \leq i \leq n$. α is 1-1 and onto because the size of A and B are both equal to m and $\mathtt{f}^{\mathbf{A}}$ and $\mathtt{f}^{\mathbf{B}}$ are both bijections on A and B respectively.

From the construction of α, it is obvious that α preserves the interpretation of $\mathtt{f}$. Moreover, since $\mathtt{g}^{\mathbf{A}}$ is the inverse of $\mathtt{f}^{\mathbf{A}}$ and $\mathtt{g}^{\mathbf{B}}$ is the inverse of $\mathtt{f}^{\mathbf{B}}$, it is again clear that α preserves the interpretation of $\mathtt{g}$.

Next, by letting $\ell = 0$ in (8) we see that α preserves the interpretation of $\mathtt{0}$. Since $\mathbf{A} \models \theta(\bar{a})$, if we choose $\ell < m$ so that $\mathtt{0}^{\mathbf{A}} + \ell = \mathtt{1}^{\mathbf{A}}$, then the formula $\mathtt{f}^{\ell}(\mathtt{0}) \approx \mathtt{1}$, as well as each $\mathtt{f}^{q}(\mathtt{0}) \not\approx \mathtt{1}$ for $0 \leq q < \ell$, must be a conjunct of $\theta(\bar{x})$: i.e., $\mathtt{0} + \ell \approx \mathtt{1}$ is a logical consequence of $\theta(\bar{x})$. Now since $\mathbf{B} \models \theta(\bar{b})$, we should have $\mathtt{0}^{\mathbf{B}} + \ell = \mathtt{1}^{\mathbf{B}}$, which means that α preserves the interpretation of $\mathtt{1}$. By the same reasoning we must have $\alpha(a_i) = \alpha(b_i)$ for every $1 \leq i \leq n$. This proves the claim.

Now from our initial supposition $\mathbf{A} \models \varphi(\bar{a})$ and the claim we just proved, we see $\mathbf{B} \models \varphi(\bar{b})$. We have shown $\mathcal{K}_m^0 \models \theta(\bar{x}) \rightarrow \varphi(\bar{x})$ as was desired. Thus $\theta(\bar{x}) \in \Theta_m(\bar{x})$ and consequently we have $\mathcal{K}_m^0 \models \varphi \rightarrow \theta_m(\varphi)$. □

2.8 Lemma *Let $\varphi(\bar{x})$ be any L^1-formula and $m \geq 2$. If $\theta \in \Theta_m(\bar{x})$ are given so that $\theta \wedge \varphi$ has a model in $\mathcal{K}_m^0$, then $\mathcal{K}_m^0 \models \theta \rightarrow \varphi$.*

Proof. Suppose that $\theta \wedge \varphi$ has a model $\langle \mathbf{A}, \bar{a}\rangle$ in $\mathcal{K}_m^0$. Then, by lemma 2.7, our supposition implies $\mathbf{A} \models \big(\bigvee \Theta_m(\varphi)\big)(\bar{a})$. If $\theta \notin \Theta_m(\varphi)$, then by lemma 2.5.$(c)$ and 2.5(d), $\mathbf{A} \not\models \theta(\bar{a})$ which is against our supposition. Therefore $\theta \in \Theta_m(\varphi)$; that is, $\mathcal{K}_m^0 \models \theta \rightarrow \varphi$. □

Next we are going to give an "infinite version" of 2.5, 2.6, 2.7 and 2.8 in 2.12, 2.13, 2.15 and 2.11 respectively. But we need some technical lemmas before that.

2.9 LEMMA *Let a sequence of variables* $\bar{x} \stackrel{\text{def}}{=} \langle x_1, \ldots, x_n \rangle$ *and integers* $k \geq 0$ *and* $m > (n+2) \cdot (k+1)$ *be given.*

(a) For $\mathbf{A} \in \mathcal{K}^0_\infty$ *and* $a_1, \ldots, a_n \in A$*, we can construct* $\mathbf{B} \in \mathcal{K}^0_m$ *and* $b_1, \ldots, b_n \in B$ *so that*

$$\mathbf{A} \models \psi(a_1, \ldots, a_n) \Leftrightarrow \mathbf{B} \models \psi(b_1, \ldots, b_n), \tag{9}$$

for all quantifier free formulas $\psi(\bar{x})$ *such that each of its atomic subformulas is of the form* $\mathtt{f}^q(x) \approx y$ *where* $q \leq k$ *and* $x, y \in \{x_1, \ldots, x_n, \mathtt{0}, \mathtt{1}\}$*.*

(b) For $\mathbf{A} \in \mathcal{K}^0_m$ *and* $a_1, \ldots, a_n \in A$*, we can construct* $\mathbf{B} \in \mathcal{K}^0_\infty$ *with* $B = \mathbb{Z}$ *and* $b_1, \ldots, b_n \in B$ *so that (9) holds.*

PROOF. Note that we can assume that the quantifier free formula ψ in (9) is in fact atomic without loss of generality.

To prove (a), let $\mathbf{A} \in \mathcal{K}^0_\infty$ and $\bar{a} \stackrel{\text{def}}{=} \langle a_1, \ldots, a_n \rangle \in A^n$ be given. To construct $\mathbf{B} \in \mathcal{K}^0_m$ and $\bar{b} \stackrel{\text{def}}{=} \langle b_1, \ldots, b_n \rangle \in B^n$ with the desired property, let B be any set of size m and let $b^1, \ldots, b^m$ be an enumeration of all elements of B. Interpret $\mathtt{f}$ by

$$\mathtt{f}^{\mathbf{B}}(b^i) \stackrel{\text{def}}{=} \begin{cases} b^{i+1}, & \text{if } 1 \leq i < m, \\ b^1, & \text{otherwise.} \end{cases} \tag{10}$$

Interpret $\mathtt{g}$ so that $\mathtt{g}^{\mathbf{B}}$ is the inverse of $\mathtt{f}^{\mathbf{B}}$. We need to interpret the two constants $\mathtt{0}, \mathtt{1}$ (i.e., assign individuals $\mathtt{0}^{\mathbf{B}} \in B$ and $\mathtt{1}^{\mathbf{B}} \in B$ to $\mathtt{0}$ and $\mathtt{1}$ respectively) and assign individuals $b_1, \ldots, b_n \in B$ to $x_1, \ldots, x_n$ respectively so that (9) holds. The assignment process is straightforward but tedious, and it goes as follows.

Define a reflexive binary relation $\preccurlyeq$ on the set $\{x_1, \ldots, x_n, \mathtt{0}, \mathtt{1}\} \stackrel{\text{def}}{=} U$ by

$$x \preccurlyeq y \stackrel{\text{def}}{\Leftrightarrow} \mathtt{f}^q(x) \approx y \in \Psi^k_{\mathbf{A},\bar{a}} \text{ for some } 0 \leq q \leq k. \tag{11}$$

Let $\leq$ be the transitive closure of $\preccurlyeq$ and let E be the symmetric transitive closure of $\preccurlyeq$.

E is an equivalence relation. Enumerate all E-classes by $U_1, \ldots, U_r$, which form a partition of U. Let s_i be the size of U_i for each $1 \leq i \leq r$. Note that $\sum_{i=1}^r s_i = n+2 = |U|$.

Observe that for $\{x, y, z\} \subseteq U_i$ with $1 \leq i \leq r$, and for $q \leq q'$,

$$\text{if } \{\mathtt{f}^q(x) \approx y, \mathtt{f}^{q'}(x) \approx z\} \subseteq \Psi^k_{\mathbf{A},\bar{a}}, \text{ then } \mathtt{f}^{q'-q}(y) \approx z \in \Psi^k_{\mathbf{A},\bar{a}}, \text{ and} \tag{12}$$

$$\text{if } \{\mathtt{f}^q(x) \approx y, \mathtt{f}^{q'}(z) \approx y\} \subseteq \Psi^k_{\mathbf{A},\bar{a}}, \text{ then } \mathtt{f}^{q'-q}(z) \approx x \in \Psi^k_{\mathbf{A},\bar{a}}. \tag{13}$$

Now by (12) and (13), for each $1 \leq i, j \leq r$ we have the implication

$$\Big(x \in U_i \text{ and } y \in U_j\Big) \Rightarrow \Big((x \leq y \text{ or } y \leq x) \text{ iff } i = j\Big), \tag{14}$$

or, in other words, E is simply the symmetric closure of $\leq$. This enables us to find an enumeration $u^1_1, u^1_2, \ldots, u^1_{s_1}$ of all elements of U_1 such that

$$u^1_1 \preccurlyeq u^1_2 \preccurlyeq \cdots \preccurlyeq u^1_{s_1}.$$

It is clear that we can choose natural numbers $q_1^1, q_2^1, \dots, q_{s_1-1}^1 \leq k$ so that

$$\mathtt{f}^{q_1^1}(u_1^1) \approx u_2^1,\ \mathtt{f}^{q_2^1}(u_2^1) \approx u_3^1,\ \dots,\ \mathtt{f}^{q_{s_1-1}^1}(u_{s_1-1}^1) \approx u_{s_1}^1 \in \Psi^k_{\mathbf{A},\bar{a}}. \tag{15}$$

Now assign

$$\begin{aligned} b^1 &\text{ to } u_1^1, \\ b^{1+q_1^1} &\text{ to } u_2^1, \\ &\vdots \\ b^{1+q_1^1+\cdots+q_{s_1-1}^1} &\text{ to } u_{s_1}^1. \end{aligned} \tag{16}$$

From (10) it is easy to verify that this assignment satisfies all the equations in (15). That is, if we let $u^{\mathbf{B}}$ denote the member of B that is assigned to $u \in U$, then $\mathbf{B} \models \mathtt{f}^{q_i^1}((u_i^1)^{\mathbf{B}}) \approx (u_{i+1}^1)^{\mathbf{B}}$ for $i = 1, \dots, s_1 - 1$. Further,

$$\mathbf{B} \models \mathtt{f}^{q_i^1+\cdots+q_{j-1}^1}((u_i^1)^{\mathbf{B}}) \approx (u_j^1)^{\mathbf{B}}, \quad \text{for all } 1 \leq i \leq j \leq s_1. \tag{17}$$

Note that an equation appearing in (17) belongs to $\Psi^k_{\mathbf{A},\bar{a}}$ if and only if $q_i^1 + \cdots + q_{j-1}^1 \leq k$, and every equation in $\Psi^k_{\mathbf{A},\bar{a}}$ appears in (17).

For inequations, among all possible inequations of the form $\mathtt{f}^q(u) \not\approx v$ where $u, v \in U_1$ and $q \leq k$, we have the following biimplications

$$\begin{aligned} &\mathtt{f}^q(u) \not\approx v \in \Psi^k_{\mathbf{A},\bar{a}} \Leftrightarrow \\ &\mathtt{f}^q(u) \approx v \text{ does not appear in (17)} \Leftrightarrow \\ &\mathbf{B} \models \mathtt{f}^q(u^{\mathbf{B}}) \not\approx v^{\mathbf{B}}. \end{aligned} \tag{18}$$

Let $Q^1 = (q_1^1 + \cdots + q_{s_1-1}^1) + (k+1)$. Note that the r.h.s. of this defining equation for Q^1 is the exponent of b in (16) minus 1 plus $(k+1)$. Also note that $Q^1 \leq s_1 \cdot (k+1)$.

Next we work on U_2. Choose an enumeration $u_1^2, \dots, u_{s_2}^2$ of all elements of U_2 and natural numbers $q_1^2, \dots, q_{s_2-1}^2$ similarly, and assign

$$\begin{aligned} b^{1+Q^1} &\text{ to } u_1^2, \\ b^{1+Q^1+q_1^2} &\text{ to } u_2^2, \\ &\vdots \\ b^{1+Q^1+q_1^2+\cdots+q_{s_2-1}^2} &\text{ to } u_{s_2}^2. \end{aligned}$$

Then we can say similar things on equations and inequations as we did for U_1: i.e., (17) and (18) modified for U_2. Moreover it is clear that the following types of inequations hold.

$$\mathbf{B} \models \mathtt{f}^q(u_i^1) \not\approx u_j^2, \quad \mathbf{B} \models \mathtt{f}^q(u_j^2) \not\approx u_i^1, \quad \text{for all } q \leq k,\ 1 \leq i \leq s_1,\ 1 \leq j \leq s_2. \tag{19}$$

Let $Q^2 = (Q^1 + q_1^2 + \cdots + q_{s_2-1}^2) + (k+1)$. Note that $Q^2 \leq (s_1 + s_2) \cdot (k+1)$. Next we work on U_3 similarly.

We continue this process until the last E-class U_r is done: i.e., until all the individual assignments to members of $\{x_1, \dots, x_n, \mathtt{0}, \mathtt{1}\}$ are done. Note that $Q^r \le |U| \cdot (k+1) = (n+2) \cdot (k+1) < m$. This completes the construction of $\mathbf{B} \in \mathcal{K}_m^0$ and $\bar{b} \in B^n$.

Now we have to verify the biimplication (9). Assume $\mathbf{A} \models \psi(\bar{a})$ where $\psi = \mathtt{f}^q(x) \approx y$, $q \le k$ and $\{x, y\} \subseteq U$. Then, by the definition of $\preccurlyeq$ in (11), we should have $\{x, y\} \subseteq U_i$ for some $1 \le i \le r$. Without loss of generality we may assume that $i = 1$. Since $x \preccurlyeq y$, we must have $x = u_i^1$ and $y = u_j^1$ for some $i \le j$. By using (17) we see that $\mathbf{B} \models \psi(\bar{b})$. We have shown $\Rightarrow$ of (9).

To prove the other direction, assume $\mathbf{A} \not\models \psi(\bar{a})$ and $\psi = \mathtt{f}^q(x) \approx y$. It suffices to show that $\mathbf{B} \not\models \psi(\bar{b})$. There are two cases to consider. In the first case, x and y belong to the same E-class. Then we assume $\{x, y\} \subseteq U_1$ without loss of generality and use (18). In the latter case, x and y belong to different E-classes. Then we use an extended version of (19). This completes the proof of (a).

Now to prove part (b) of this lemma, suppose that $\mathbf{A} \in \mathcal{K}_m^0$ and $\bar{a} \in A^n$ are given. Enumerate the elements of A as $a^1, \dots, a^m$ so that $\mathtt{f}^{\mathbf{A}}(a^i) = a^{i+1}$, $(1 \le i < m)$ and $\mathtt{f}^{\mathbf{A}}(a^m) = a^1$. Choose $1 \le i_0, i_1 \le m$ so that $a^{i_0} = \mathtt{0}^{\mathbf{A}}$ and $a^{i_1} = \mathtt{1}^{\mathbf{A}}$. Choose $1 \le i(1), \dots, i(n) \le m$ so that $a_j = a^{i(j)}$ for all $1 \le j \le n$. Let $i_{\max}$ be the largest member of the set $\{i(1), \dots, i(n), i_0, i_1\}$. Since $m > (n+2) \cdot (k+1)$, we can assume that the enumeration $a_1, \dots, a_n$ has been done so that

$$i_{\max} + k < m. \tag{20}$$

Now we construct a member $\mathbf{B} \in \mathcal{K}_\infty^0$ and $\bar{b} \in B^n$ as follows: take any denumerable set and enumerate as $\{b^i \mid i \in \mathbb{Z}\}$. Interpret $\mathtt{f}$ so that $\mathtt{f}^{\mathbf{B}}(b^i) = b^{i+1}$ for each $i \in \mathbb{Z}$. Interpret $\mathtt{g}$ so that $\mathtt{g}^{\mathbf{B}}$ is the inverse of $\mathtt{f}^{\mathbf{B}}$. Let $\mathtt{0}^{\mathbf{B}} = b^{i_0}$ and let $\mathtt{1}^{\mathbf{B}} = b^{i_1}$. Let $b_j = b^{i(j)}$ for each $1 \le j \le n$.

We claim that the biimplication (9) holds. To prove the claim it suffices to prove $\Rightarrow$ of (9) for the following two cases: $\psi = \mathtt{f}^q(x) \not\approx y$ case and $\psi = \mathtt{f}^q(x) \approx y$ case. The first case is trivial from the construction. For the second case, use (20) and the fact that $q \le k$. This completes the proof of (b). □

2.10 LEMMA *Let a sequence of variables* $\bar{x} \overset{\text{def}}{=} \langle x_1, \dots, x_n \rangle$ *and an integer* $k \ge 0$ *be given. Suppose that* $\varphi(\bar{x})$ *is a quantifier free formula such that each of its atomic subformulas is of the form* $\mathtt{f}^q(x) \approx y$ *where* $q \le k$ *and* $\{x, y\} \subseteq \{x_1, \dots, x_n, \mathtt{0}, \mathtt{1}\}$. *Then for each* $\theta \in \Theta_\infty^k(\bar{x})$ *such that* $\theta \wedge \varphi$ *is consistent, we have*

$$\models \theta \to \varphi. \tag{21}$$

As a corollary, if φ_1 *and* φ_2 *are formulas that satisfy the same condition as* φ, *then*

$$\Theta_\infty^k(\varphi_1 \vee \varphi_2) = \Theta_\infty^k(\varphi_1) \cup \Theta_\infty^k(\varphi_1), \tag{22}$$

and

$$\Theta_\infty^k(\neg\varphi_1) = \Theta_\infty^k(\bar{x}) - \Theta_\infty^k(\varphi_1). \tag{23}$$

PROOF. Assume the hypotheses. We prove the conclusion (21) by induction on the complexity of φ: we give proofs for the cases φ is an equation, an inequation, a disjunction of two formulas and then finally a conjunction of two formulas: that will suffice, because every quantifier free formula is logically equivalent to one in conjunctive normal form.

Suppose that $\varphi = \mathtt{f}^q(x) \approx y$. Then θ must have φ among its conjuncts, for otherwise it would have $\mathtt{f}^q(x) \not\approx y \ \big(= \neg\varphi\big)$ instead of $\mathtt{f}^q(x) \approx y$ which contradicts the consistency of $\theta \wedge \varphi$. Thus (21) holds. The case when φ is an inequation can be handled by the same reasoning.

Next suppose that $\varphi = \varphi_1 \vee \varphi_2$ for some formulas φ_1 and φ_2. Then $\theta \wedge \varphi_i$ is consistent for $i = 1$ or for $i = 2$. In the first case we have $\models \theta \to \varphi_1$ by the induction hypothesis, and consequently (21) holds. The second case is similar.

The proof for the final case $\theta = \varphi_1 \wedge \varphi_2$ is equally easy and hence will be omitted.

For the proof of (22), observe that for $\theta \in \Theta^k_\infty(\bar{x})$, the following series of biimplications hold:

$$\begin{aligned}
\theta \in \Theta^k_\infty(\varphi_1 \vee \varphi_2) &\leftrightarrow \theta \wedge (\varphi_1 \vee \varphi_2) \text{ is consistent} \\
&\leftrightarrow \theta \wedge \varphi_1 \text{ is consistent or } \theta \wedge \varphi_2 \text{ is consistent} \\
&\leftrightarrow \models \theta \to \varphi_1 \text{ or } \models \theta \to \varphi_2 \\
&\leftrightarrow \theta \in \Theta^k_\infty(\varphi_1) \text{ or } \theta \in \Theta^k_\infty(\varphi_2) \\
&\leftrightarrow \theta \in \Theta^k_\infty(\varphi_1) \cup \Theta^k_\infty(\varphi_2).
\end{aligned}$$

For the proof of (23), observe that for $\theta \in \Theta^k_\infty(\bar{x})$, the following series of biimplications hold:

$$\begin{aligned}
\theta \in \Theta^k_\infty(\neg\varphi_1) &\leftrightarrow \theta \wedge \neg\varphi_1 \text{ is consistent} \\
&\leftrightarrow \not\models \theta \to \varphi_1 \\
&\leftrightarrow \theta \notin \Theta^k_\infty(\varphi_1).
\end{aligned}$$

□

Note that, in above lemma, we can replace $\Theta^k_\infty(\bar{x}),\ k \geq 0$ by $\Theta_m(\bar{x}),\ m \geq 2$.

2.11 LEMMA *Let $x_1, \ldots, x_n$ and z be $n+1$ distinct variables and let $k \geq 0$. If two L^1-formulas $\theta(\bar{x}) \in \Theta^{2k}_\infty(\bar{x})$ and $\psi(\bar{x}, z) \in \Theta^k_\infty(\bar{x}, z)$ are given so that $\theta(\bar{x}) \wedge \exists z\psi(\bar{x}, z)$ has a model in $\mathcal{K}^0_\infty$, then*

$$\mathcal{K}^0_{>(n+2)\cdot(2k+1)} \models \theta(\bar{x}) \to \exists z\psi(\bar{x}, z). \tag{24}$$

PROOF. Let $\theta(\bar{x})$ and $\psi(\bar{x}, z)$ be given as in the hypothesis. Then there exist $\mathbf{A} \in \mathcal{K}^0_\infty$ and $\langle a_1, \ldots, a_n, a_{n+1}\rangle \in A^{n+1}$ such that

$$\mathbf{A} \models \theta(\bar{a}) \wedge \psi(\bar{a}, a_{n+1}). \tag{25}$$

Observe that the satisfaction relation (25) implies that each conjunct of ψ not involving the variable z is also a conjunct of θ, and if $\mathtt{f}^q(x) \approx y$ (resp. $\mathtt{f}^q(x) \not\approx y$) is a conjunct of θ then it is also a conjunct of ψ if and only if $q \leq k$.

Now to prove (24), let $m > (n+2)\cdot(2k+1)$ and let $\mathbf{B} \in \mathcal{K}^0_m$ and $\langle b_1, \dots, b_n\rangle \in B^n$ be given, and assume that $\mathbf{B} \models \theta(\bar{b})$. We want to show that $\mathbf{B} \models \psi(\bar{b}, b_{n+1})$ for some $b_{n+1} = z^{\langle \mathbf{B}, \bar{b}, b_{n+1}\rangle} \in B$. Note that, by an observation made in the previous paragraph, in order to verify $\mathbf{B} \models \psi(\bar{b}, b_{n+1})$, we only have to check those conjuncts ψ' of ψ of the form

$$\mathtt{f}^q(u) \approx z,\ \mathtt{f}^q(z) \approx u,\ \mathtt{f}^q(u) \not\approx z,\ \mathtt{f}^q(z) \not\approx u \tag{26}$$

where $u \in \{x_1, \dots, x_n, \mathtt{0}, \mathtt{1}\}$. Observe that we have excluded the possibility $u = z$ in the equations and inequations in (26) because, for equations, this would force either $\psi' = z \approx z$ which is trivially satisfied in $\langle \mathbf{B}, \bar{b}, b_{n+1}\rangle$ for any choice of $b_{n+1} \in B$, or $\psi' = \mathtt{f}^q(z) \approx z$ for some $0 < q \le k$, and this ψ' cannot exist as a conjunct of ψ since $\mathbf{A}$ has no finite cycle. And for inequations, $u = z$ would imply $\psi' = \mathtt{f}^q(z) \not\approx z$ for some $0 < q \le k$ which is surely satisfied in $\mathbf{B} \in \mathcal{K}^0_{>m}$.

We consider the following 2 cases and complete the proof by finding suitable $b_{n+1} \in B$ in each case.

Case 1. Every conjunct of ψ involving the variable z is an inequation, except the trivial case $z \approx z$. We want to show

$$(\mathtt{f}^{\mathbf{B}})^q(b) \neq b_{n+1},\ (\mathtt{f}^{\mathbf{B}})^q(b_{n+1}) \neq b, \quad \text{for all } b \in \{b_1, \dots, b_n, 0, 1\},\ 0 \le q \le k.$$

Thus it suffices to find $b_{n+1} \in B$ such that

$$\begin{aligned} b \notin \{&(\mathtt{f}^{\mathbf{B}})^{-k}(b_{n+1}), (\mathtt{f}^{\mathbf{B}})^{-k+1}(b_{n+1}), \dots, (\mathtt{f}^{\mathbf{B}})^{-1}(b_{n+1}), b_{n+1}, \\ &(\mathtt{f}^{\mathbf{B}})^{1}(b_{n+1}), (\mathtt{f}^{\mathbf{B}})^{2}(b_{n+1}), \dots, (\mathtt{f}^{\mathbf{B}})^{k}(b_{n+1})\} \text{ for all } b \in \{b_1, \dots, b_n, 0, 1\}. \end{aligned}$$

There should be no problem in finding such b_{n+1} because the size of the universe B is $> m$ which is equal to $(n+2)\cdot(2k+1)$.

Case 2. ψ has a conjunct involving the variable z which is an equation other than the trivial $z \approx z$. We will assume that the equation is of the form $\mathtt{f}^q(z) \approx v$ for some $v \in \{x_1, \dots, x_n, \mathtt{0}, \mathtt{1}\}$ and some $q \le k$, because the case $v = z$ is plainly out of question as was pointed out before, and the other case $\mathtt{f}^q(v) \approx z$ can be handled in a completely symmetric manner. Assume, without loss of generality, that $v = x_1$. Then, from (25), we have

$$a_{n+1} + q = a_1. \tag{27}$$

Pick $b_{n+1} \in B$ so that $b_{n+1} + q = b_1$.

Let ψ' be a conjunct of ψ. We want to show $\mathbf{B} \models \psi'(\bar{b}, b_{n+1})$, where ψ' is one of the four types shown in (26).

Suppose that $\psi' = \mathtt{f}^{q'}(u) \approx z$ for some $u \in \{x_1, \dots, x_n, \mathtt{0}, \mathtt{1}\}$ and some $q' \le k$. We can assume that $u \neq x_1$ for otherwise $\mathbf{A}$ would have a finite cycle (of size $\le q+q'$) contradicting our choice of $\mathbf{A} \in \mathcal{K}^0_\infty$. We will assume without loss of generality that $u = x_2$. We are going to show that $b_2 + q' = b_{n+1}$, which would imply $\mathbf{B} \models \mathtt{f}^{q'}(x_2) \approx z$ as is desired.

First note that $a_2 + q' = a_{n+1}$. Combining this with (27), we get $a_2 + (q' + q) = a_1$. Hence, by (25) and the fact that $q' + q \leq 2k$, $\mathtt{f}^{q'+q}(x_2) \approx x_1$ must be a conjunct of θ. So we have $b_2 + (q' + q) = b_1$. But $b_{n+1} + q = b_1$ by our choice of b_{n+1}. Therefore $b_2 + q' = b_1$. This completes the proof for the case when $\psi' = \mathtt{f}^{q'}(u) \approx z$.

Next suppose that $\psi' = \mathtt{f}^{q'}(z) \approx u$ for some $u \in \{x_1, \ldots, x_n, \mathtt{0}, \mathtt{1}\}$ and some $q' \leq k$. We will assume $u = x_2$ without loss of generality as before. We will also assume without loss of generality that $q \geq q'$, because the other case $q' \geq q$ case can be handled symmetrically. We are going to show that $b_{n+1} + q' = b_2$, which would imply $\mathbf{B} \models \mathtt{f}^{q'}(z) \approx x_2$ as is desired.

First note that $a_{n+1} + q' = a_2$. Combining this with (27), we get $a_2 + (q - q') = a_1$. Hence, by (25), $\mathtt{f}^{q-q'}(x_2) \approx x_1$ must be a conjunct of θ. So we have $b_2 + (q - q') = b_1$. But $b_{n+1} + q = b_1$ by our choice of b_{n+1}. Therefore $b_{n+1} + q' = b_2$. This completes the proof for the case when $\psi' = \mathtt{f}^{q'}(z) \approx u$.

Next suppose that $\psi' = \mathtt{f}^{q'}(u) \not\approx z$ for some $u \in \{x_1, \ldots, x_n, \mathtt{0}, \mathtt{1}\}$ and some $q' \leq k$. The cases when $u = z$ or $u = x_1$ can be trivially handled. So we will assume without loss of generality that $u = x_2$. We are going to show that $b_2 + q' \neq b_{n+1}$, which would imply $\mathbf{B} \models \mathtt{f}^{q'}(x_2) \not\approx z$ as is desired.

First note that $a_2 + q' \neq a_{n+1}$. Combining this with (27), we get $a_2 + (q' + q) \neq a_1$. Hence, by (25) and the fact that $q' + q \leq 2k$, $\mathtt{f}^{q'+q}(x_2) \not\approx x_1$ must be a conjunct of θ. So we have $b_2 + (q' + q) \neq b_1$. But $b_{n+1} + q = b_1$ by our choice of b_{n+1}. Therefore $b_2 + q' \neq b_{n+1}$. This completes the proof for the case when $\psi' = \mathtt{f}^{q}(u) \not\approx z$.

The proof for the final case when $\psi' = \mathtt{f}^{q'}(z) \not\approx u$ is omitted because it is as easy as above. □

2.12 Lemma *Given* $k \geq 0$ *and* $\Theta^k_\infty(\bar{x})$,

(a) $\Theta^k_\infty(\bar{x})$ *is finite,*

(b) Each $\theta \in \Theta^k_\infty(\bar{x})$ *has a model in* $\mathcal{K}^0_\infty$ *and also in* $\mathcal{K}^0_m$ *for each* $m > (n+2) \cdot (k+1)$,

(c) $\mathcal{K}^0_{>(n+2)\cdot(k+1)} \models \bigvee \Theta^k_\infty(\bar{x})$,

(d) $\mathcal{K}^0_{>(n+2)\cdot(k+1)} \models \neg(\theta \wedge \theta')$ *for all* $\theta \neq \theta' \in \Theta^k_\infty(\bar{x})$.

(e) For $\theta \in \Theta^k_\infty(\bar{x})$, $\mathbf{A} \in \mathcal{K}^0_\infty$ *and* $\bar{a} \in A^n$, $\mathbf{A} \models \theta(\bar{a})$ *iff* $\theta = \bigwedge \Psi^k_{\mathbf{A},\bar{a}}$.

Proof. (a) is obvious.

To prove (b), let $\theta(\bar{x}) \in \Theta^k_\infty(\bar{x})$ be given. From the definition of $\Theta^k_\infty(\bar{x})$, it is obvious that $\theta(\bar{x})$ has a model in $\mathcal{K}^0_\infty$: i.e., $\mathbf{A} \models \theta(\bar{a})$ for some $\mathbf{A} \in \mathcal{K}^0_\infty$ and $\bar{a} \in A^n$ such that $\theta(\bar{x}) = \bigwedge \Psi^k_{\mathbf{A},\bar{a}}$. Now apply lemma 2.9.$(a)$ to get a model of $\theta(\bar{x})$ in $\mathcal{K}^0_m$.

To prove (c), let $\mathbf{A} \in \mathcal{K}^0_{>(n+2)\cdot(k+1)}$ and $\bar{a} \in A^n$ be given. If $\mathbf{A} \in \mathcal{K}^0_\infty$, then $\mathbf{A} \models \theta(\bar{a})$ with $\theta(\bar{x}) = \bigwedge \Psi^k_{\mathbf{A},\bar{a}} \in \Theta^k_\infty(\bar{x})$. If $\mathbf{A} \in \mathcal{K}^0_m$, then apply lemma 2.9.$(b)$.

The proofs of (d) and (e) are exactly the same as those of lemma 2.5.(d) and 2.5.(e) respectively. □

2.13 DEFINITION Let $\varphi(x_1, \dots, x_n)$ be an L^1-formula in which all the atomic subformulas are of the form $\mathtt{f}^q(x) \approx y$ where $\{x, y\} \subseteq \{x_1, \dots, x_n, \mathtt{0}, \mathtt{1}\}$ and $q \geq 0$.

(*a*) For each φ, we recursively associate a natural number k_φ, called the *level* of φ as follows:

$$\begin{aligned} k_{\mathtt{f}^q(x)\approx y} &\stackrel{\text{def}}{=} q, \\ k_{\neg\varphi} &\stackrel{\text{def}}{=} k_\varphi, \\ k_{\varphi\wedge\psi} &\stackrel{\text{def}}{=} \max(k_\varphi, k_\psi), \\ k_{\exists x\varphi} &\stackrel{\text{def}}{=} 2 \cdot k_\varphi. \end{aligned}$$

(*b*) For each integer $k \geq 0$, we let

$$\begin{aligned} \Theta^k_\infty(\varphi) &\stackrel{\text{def}}{=} \{\theta \in \Theta^k_\infty(\bar{x}) \mid \mathcal{K}^0_\infty \models \theta \to \varphi\}, \text{ and let} \\ \theta^k_\infty(\varphi) &\stackrel{\text{def}}{=} \bigvee \Theta^k_\infty(\varphi). \end{aligned}$$

◀

Note that $\mathcal{K}^0_\infty$ can easily be shown to be decidable by imitating the proof of lemma 2.3. Thus, from lemma 2.12.(*a*), the process $\langle\varphi, k\rangle \mapsto \theta^k_\infty(\varphi)$ is effective.

We present some easily proved but useful facts below.

2.14 FACT

(*a*) When we deal with formulas for $\mathcal{K}^0$, we may often assume that the symbol $\mathtt{g}$ does not occur in φ because f and g are inverses to each other in any model of T^0: if a term $\mathtt{g}(t)$ occurs in φ, then we replace this occurrence of $\mathtt{g}(t)$ by a new variable x to get φ'. Then φ is equivalent to $\exists x\big((\mathtt{f}(x) \approx t) \wedge \varphi'\big)$ in all models of T^0. We repeat this to eliminate all occurrences of $\mathtt{g}$. Also we may assume that all atomic subformulas of φ are of the form $\mathtt{f}^k(t) \approx s$ where t and s are variables or constants because f is a bijection and equality is symmetric.

(*b*) By lemma 2.12.(*d*), if Θ_0 and Θ_1 are subsets of $\Theta^k_\infty(\bar{x})$ such that $\Theta_0 \cap \Theta_1 = \varnothing$ then $\models \bigvee\Theta_0 \to \neg\bigvee\Theta_1$. On the other hand, by lemma 2.12.(*c*), if $\Theta_0 \cup \Theta_1 = \Theta^k_\infty(\bar{x})$, then $\mathcal{K}^0_{>(n+2)\cdot(k+1)} \models \neg\bigvee\Theta_1 \to \bigvee\Theta_0$. Combining these two, we have $\mathcal{K}^0_{>(n+2)\cdot(k+1)} \models \bigvee\Theta_0 \leftrightarrow \neg\bigvee\Theta_1$ whenever Θ_0 and Θ_1 partition $\Theta^k_\infty(\bar{x})$.

(*c*) In fact (*b*) holds when we replace $\Theta^k_\infty(\bar{x})$ by $\Theta_m(\bar{x})$ by a similar argument. So if Θ_0 and Θ_1 partition $\Theta_m(\bar{x})$, then $\mathcal{K}^0_m \models \bigvee\Theta_0 \leftrightarrow \neg\bigvee\Theta_1$. □

2.15 LEMMA *Given any L^1-formula $\varphi(x_1, \dots, x_n)$ such that each of its atomic subformulas is of the form $\mathtt{f}^q(x) \approx y$ where $q \leq k$ and $x, y \in \{x_1, \dots, x_n, \mathtt{0}, \mathtt{1}\}$, and given any integer $k \geq k_\varphi$, we have*

$$\mathcal{K}^0_{>(n+2)\cdot(k+1)} \models \varphi \leftrightarrow \theta^k_\infty(\varphi).$$

PROOF. The proof proceeds by induction on the complexity of φ.

Suppose that φ is atomic. In this case, by lemma 2.9.(b), it suffices to show

$$\mathcal{K}^0_\infty \models \varphi \leftrightarrow \theta^k_\infty(\varphi). \tag{28}$$

$\leftarrow$ of (28) is obvious from the construction of $\theta^k_\infty(\varphi)$. To prove $\rightarrow$ of (28), suppose that $\mathbf{A} \in \mathcal{K}^0_\infty$ and $\bar{a} \in A^n$ are given so that $\mathbf{A} \models \varphi(\bar{a})$. We want to show $\mathbf{A} \models \theta^k_\infty(\varphi)(\bar{a})$.

Let $\theta(\bar{x}) = \bigwedge \Psi^k_{\mathbf{A},\bar{a}}$. Obviously $\mathbf{A} \models \theta(\bar{a})$. In order to show $\mathbf{A} \models \theta^k_\infty(\varphi)(\bar{a})$, it suffices to show that $\theta(\bar{x}) \in \Theta^k_\infty(\varphi)$, or equivalently, $\mathcal{K}^0_\infty \models \theta(\bar{x}) \rightarrow \varphi(\bar{x})$ because $\theta(\bar{x}) \in \Theta^k_\infty(\bar{x})$. But this is an immediate consequence of lemma 2.10 and the fact that $\mathbf{A} \models (\theta \wedge \varphi)(\bar{a})$. This completes the proof for the case when φ is atomic.[2]

If $\varphi = \neg\varphi_1$ for some formula φ_1, then we have a series of biimplications

$$\begin{aligned}
\mathcal{K}^0_{>(n+2)\cdot(k+1)} \models \varphi &\leftrightarrow \neg\varphi_1 && \text{by definition,} \\
&\leftrightarrow \neg\theta^k_\infty(\varphi_1) && \text{by induction hypothesis,} \\
&\leftrightarrow \bigvee \left(\Theta^k_\infty(\bar{x}) - \Theta^k_\infty(\varphi_1)\right) && \text{by fact 2.14.}(b), \\
&\leftrightarrow \bigvee \Theta^k_\infty(\neg\varphi_1) && \text{by lemma 2.10.(23),} \\
&\leftrightarrow \theta^k_\infty(\varphi) && \text{by definition.}
\end{aligned} \tag{29}$$

If $\varphi = \varphi_1 \vee \varphi_2$ for some formulas φ_1 and φ_2, then use lemma 2.10.(22) instead of 2.10.(23).

It remains to handle the case $\varphi(\bar{x}) = \exists y \psi(\bar{x}, y)$. We let $k \geq k_\varphi$ and let $M = (n+2) \cdot (k+1)$. We are going to show

$$\mathcal{K}^0_{>M} \models \varphi \leftrightarrow \theta^k_\infty(\varphi). \tag{30}$$

$\leftarrow$ of (30) is again obvious. To prove $\rightarrow$, suppose $\mathbf{A} \in \mathcal{K}^0_{>M}$ and $\bar{a} \in A^n$ satisfy $\mathbf{A} \models \varphi(\bar{a})$. We want to show that $\mathbf{A} \models \theta^k_\infty(\varphi)$. It suffices to show that there exists $\theta \in \Theta^k_\infty(\bar{x})$ such that $\mathcal{K}^0_{>M} \models \theta \rightarrow \varphi$ and $\mathbf{A} \models \theta(\bar{a})$. Take $\theta = \Psi^k_{\mathbf{A},\bar{a}}$. We want to show $\mathcal{K}^0_{>M} \models \theta \rightarrow \exists y \psi(\bar{x}, y)$. Since $\mathcal{K}^0_{>M} \models \psi(\bar{x}, y) \leftrightarrow \theta^{k_\psi}_\infty(\psi)(\bar{x}, y)$ by induction and $\exists$ operator distributes over $\vee$, we may assume without loss of generality that $\psi \in \Theta^{k_\psi}_\infty(\bar{x}, y)$. Now since $k \geq k_\varphi = 2 \cdot k_\psi$, we can apply lemma 2.11 to complete the proof. $\square$

2.16 THEOREM *T^0 admits elimination of quantifiers: i.e., given an L^1-formula $\varphi(\bar{x})$, we can effectively find a quantifier-free formula $\varphi^*(\bar{x})$ such that $T^0 \vdash \varphi \leftrightarrow \varphi^*$ and each free variable of φ^* is a free variable of φ. Thus $T^1 \supseteq T^0$ admits elimination of quantifiers too.*

PROOF. Let an L^1-formula $\varphi(\bar{x})$ be given. First, by fact 2.14.(a), assume without loss of generality that all atomic subformulas of $\varphi(\bar{x})$ are of the form $\mathtt{f}^q(x) \approx y$ where $x, y \in \{x_1, \ldots, x_n, \mathtt{0}, \mathtt{1}\}$ and $q \in \omega$.

Now let $M = (n+2) \cdot (k_\varphi + 1)$ and let

$$\varphi^* \stackrel{\text{def}}{=} \bigvee \{\mathtt{Cycle}_m \wedge \theta_m(\varphi) \mid 1 < m \leq M\} \vee \left(\mathtt{Cycle}_{>M} \wedge \theta^{k_\varphi}_\infty(\varphi)\right). \tag{31}$$

Then $\mathcal{K}^0 \models \varphi \leftrightarrow \varphi^*$ is easily proved using lemma 2.7 and lemma 2.15. $\square$

[2] Actually this proof works for the case when φ is quantifier free.

2.17 Theorem *The universally axiomatizable theory T^1 is decidable but not finitely decidable.*

Proof. Let an L^1-sentence φ be given, and let φ^* be the formula in (31). By theorem 2.16, it suffices to show that we can effectively determine whether φ^* has a model $\mathbf{A} \in \mathcal{K}^0$ in which

$$\big(0 + n = 1\big) \Rightarrow \big(1 + h(n) = 0\big) \text{ for all } n \in \omega. \tag{$*$}$$

We examine the possibility of the existence of such a model of φ^* for $m = 2, \dots, M \stackrel{\text{def}}{=} (0 + 2) \cdot (k_\varphi + 1)$ and $m > M$.

Let $m \leq M$ and let ψ be a disjunct of $\theta_m(\varphi)$. We want to know whether ψ has model in $\mathcal{K}^0_m$ in which $(*)$ holds. Recall that ψ is basically a conjunction of formulas of the form

$$\mathtt{f}^q(\mathtt{0}) \approx \mathtt{1},\ \mathtt{f}^q(\mathtt{0}) \not\approx \mathtt{1},\ \mathtt{f}^q(\mathtt{1}) \approx \mathtt{0},\ \mathtt{f}^q(\mathtt{1}) \not\approx \mathtt{0}, \quad q < m. \tag{32}$$

Note that exactly two equations occur as conjuncts of ψ—one is of the form $\mathtt{f}^q(\mathtt{0}) \approx \mathtt{1}$ and the other is of the form $\mathtt{f}^q(\mathtt{1}) \approx \mathtt{0}$, and the rest are all inequations. Now check, for those n such that $1 \leq n < m$ and $n + h(n) = m$, whether the formulas $\mathtt{f}^n(\mathtt{0}) \approx \mathtt{1}$ and $\mathtt{f}^{h(n)}(\mathtt{1}) \approx \mathtt{0}$ occur as conjuncts of ψ. If this is the case then ψ has the desired model for m. Otherwise ψ does not.

After having checked all these for $m = 2, \dots, M$, if ψ still does not have a desired model, then we check whether the model exists in $\mathcal{K}^0_{>M} \supseteq \mathcal{K}^0_\infty$. Let ψ be a disjunct of $\Theta^{k_\varphi}_\infty(\varphi)$. It is again a conjunction of formulas of the form (32) except that $q \leq k_\varphi$ instead of $q < m$. But this time the number of equations of the form $\mathtt{f}^q(\mathtt{0}) \approx \mathtt{1}$ may be 0 as well as 1. The same is true for equations of the form $\mathtt{f}^q(\mathtt{1}) \approx \mathtt{0}$.

If an equation of the form $\mathtt{f}^n(\mathtt{0}) \approx \mathtt{1}$ occurs as a conjunct of ψ, then ψ has a desired model if and only if

$$n + h(n) > M \text{ and } \psi \text{ has no conjunct of the form } f^k(\mathtt{1}) \approx \mathtt{0}$$

because $M = (n + 2) \cdot (k + 1) > 2k_\varphi$.

Otherwise ψ has a desired model $\mathbf{A}$ in $\mathcal{K}^0_\infty$: that is, let $A = \mathbb{Z}$ and let $\mathtt{0}^{\mathbf{A}} = 0$ and $\mathtt{1}^{\mathbf{A}} = -k$ for some $k > 0$ where k is the integer determined as follows: if an equation of the form $\mathtt{f}^q(\mathtt{1}) \approx \mathtt{0}$ exists as a conjunct of ψ, then $k = q$. Otherwise, k can be arbitrary, say $k = 1$. ☐

2.18 Corollary

(a) For each $m > 1$, $\mathcal{K}^1_m$ is decidable.

(b) $\mathcal{K}^1_\infty$ is decidable.

Proof. To prove (a), just note that the theory of $\mathcal{K}^1_m$ is a finite extension of the theory of $\mathcal{K}^1$. To prove (b), imitate (a part of) the proof of lemma 2.3. ☐

So we have got a class $\mathcal{K}^1$ of first-order structures having the following properties:

(a) $\mathcal{K}^1$ is decidable,

(b) $\mathcal{K}^1_{\text{fin}}$ is undecidable,

(c) the language of $\mathcal{K}^1$ is finite and contains no predicate symbols,

(d) $\mathcal{K}^1$ is universally axiomatizable.

We want to strengthen this result to replace (d) by

(d') $\mathcal{K}^1$ is equationally axiomatizable.

The rest of this paper is devoted to finding such an example.

3 The Construction

Let $\mathtt{t}$ be a ternary operation symbol and let δ be the following sentence, which says that t is the *discriminator operation*.

$$\delta \stackrel{\text{def}}{=} \forall x, y, z, w \Big(\mathtt{t}(x, y, z) \approx w \leftrightarrow \big((x \approx y \wedge w \approx z) \vee (x \not\approx y \wedge w \approx x)\big)\Big). \tag{33}$$

For an algebra $\mathbf{A}$ not having $\mathtt{t}$ as a basic operation symbol, we let $\mathbf{A}^{\mathtt{t}}$ be the algebra $\langle \mathbf{A}, t \rangle$ in the expanded language where t is the discriminator operation for A; i.e., $\mathbf{A}^{\mathtt{t}} \models \delta$. For a class $\mathcal{K}$ of similar algebras not having $\mathtt{t}$ as a basic operation symbol, we define $\mathcal{K}^{\mathtt{t}}$ to be the collection $\{\mathbf{A}^{\mathtt{t}} \mid \mathbf{A} \in \mathcal{K}\}$. Now we define our variety $\mathcal{V}$ which is decidable but not finitely decidable.

3.1 Definition

(a) We fix a language L that we will use throughout this paper as follows:

$$L \stackrel{\text{def}}{=} L^1 \cup \{\mathtt{t}\} = \{\mathtt{t}, \mathtt{f}, \mathtt{g}, \mathtt{0}, \mathtt{1}\}.$$

(b) Define an L-class $\mathcal{W}$ by

$$\mathcal{W} = (\mathcal{K}^1)^{\mathtt{t}},$$

where $\mathcal{K}^1$ is defined in definition 2.1, and then define a variety $\mathcal{V}$ by

$$\mathcal{V} = \mathsf{HSP}(\mathcal{W}).$$

◀

Note that $\mathcal{W}$ is a universal class axiomatized by the axioms of $\mathcal{K}^1$ plus the discriminator axiom δ. Throughout this paper the variety $\mathcal{V}$ and the universal class $\mathcal{W}$ will be fixed.

3.2 Theorem *$\mathcal{V}$, which is defined in 3.1, is finitely undecidable.*

PROOF. A well-known theorem of Bulman-Fleming, Keimel and Werner says that if $\mathcal{K}$ is a class of similar algebras having a common discriminator term $\mathtt{t}$ such that $\mathcal{K} \models \delta$, where δ is the *discriminating axiom* defined in (33), then $\mathsf{HSP}(\mathcal{K}) = \mathsf{I}\Gamma^{\mathsf{a}}\mathsf{SP}_{\mathsf{U}}(\mathcal{K})$ and $\mathsf{ISP}_{\mathsf{U}}(\mathcal{K}) = \{\mathbf{A} \in \mathsf{HSP}(\mathcal{K}) \mid \mathbf{A} \models \delta\}$ [BS81]. Now since $\mathcal{W}$ is a universal class, we have

$$\begin{aligned} \mathcal{W} &= \mathsf{ISP}_{\mathsf{U}}(\mathcal{W}) = \{\mathbf{A} \in \mathsf{HSP}(\mathcal{W}) \mid \mathbf{A} \models \delta\}, \text{ and} \\ \mathcal{V} &= \mathsf{I}\Gamma^{\mathsf{a}}(\mathcal{W}). \end{aligned} \tag{34}$$

Since $\mathcal{W}_{\text{fin}} = (\mathcal{K}^1_{\text{fin}})^{\mathtt{t}}$ and $\mathcal{K}^1$ is finitely undecidable $\mathcal{W}$ cannot be finitely decidable. By (34), we have

$$\mathcal{V}_{\text{fin}} \models \delta \to \sigma \;\Leftrightarrow\; \mathcal{W}_{\text{fin}} \models \sigma$$

for all L-sentence σ. Thus $\mathcal{V}_{\text{fin}}$ must be undecidable. $\square$

Proving the decidability of $\mathcal{V}$ is much harder. Basically we use the fact that the class $\mathcal{B}$ of all Boolean algebras with denumerably many constant symbols is decidable [Tar49]. Let L_B be the language of $\mathcal{B}$. We will prove the decidability of $\mathcal{V}$ by constructing an effective map $\varphi \mapsto \varphi^{\#}$ from the set of L-sentences into the set of L_B-sentences in such a way that $\big(\varphi$ has a model in $\mathcal{V}\big) \Leftrightarrow \big(\varphi^{\#}$ has a model in $\mathcal{B}\big)$.

4 A Feferman-Vaught Type Theorem

4.1 DEFINITION (*a*) We define the language L_B of Boolean algebras by

$$L_B \overset{\text{def}}{=} \{\cap, \cup, -, \mathtt{0}, \mathtt{1}\} \cup \{\mathtt{C}_m \mid m > 1\},$$

where $\cap$ and $\cup$ are binary operation symbols, $-$ is a unary operation symbol, $\mathtt{0}, \mathtt{1}$ and $\mathtt{C}_m$'s are constant symbols. By abuse of notation we may use $-$ as a binary operation symbol defined by $t_1 - t_2 \overset{\text{def}}{=} t_1 \cap (-t_2)$ where t_1 and t_2 are L_B-terms.

We assume that L_B has an additional binary predicate symbol $\subseteq$ with the meaning $t_1 \subseteq t_2 \Leftrightarrow t_1 \cap t_2 = t_1$: that is, we abbreviate the formula $t_1 \cap t_2 \approx t_1$ as $t_1 \subseteq t_2$.

For each $m > 1$, we abbreviate the term $\mathtt{1} - \bigcup_{i=2}^{m} \mathtt{C}_i$ as $\mathtt{C}_{>m}$.

(*b*) Let $\mathcal{B}$ be the class of all L_B-structures $\mathbf{B} \overset{\text{def}}{=} \langle B, \cap, \cup, \dots \rangle$ such that $\langle B, \cap, \cup, -, 0, 1\rangle$ is a Boolean algebra. For each $M \geq 2$, let $\mathcal{B}_M \subseteq \mathcal{B}$ be the class of all members $\mathbf{B}$ of $\mathcal{B}$ in which $\mathbf{B} \models \mathtt{C}_m \approx \mathtt{0}$ for every $m > M$.

(*c*) For each L-formula δ, we assign a unique individual variable Y_δ of L_B. If Δ is a set of L-formulas and if an L_B-formula φ^* has all its free variables among $\{Y_\delta \mid \delta \in \Delta\}$, then we may write φ^* as $\varphi^*(Y_\delta \mid \delta \in \Delta)$.

(*d*) For each $n, k \in \omega$, we define a set of L-formulas:

$$\Delta(n,k) \overset{\text{def}}{=} \bigcup_{m=2}^{M} \Theta_m(\bar{x}) \cup \Theta^k_\infty(\bar{x}), \text{ where } M = (n+2)\cdot(k+1).$$

(e) For pairwise distinct variables $x_1, \dots, x_n$ and y, we let $\mathtt{Part}(x_1, \dots, x_n, y)$ be a (fixed) quantifier free L_B formula saying that $\{x_1, \dots, x_n\}$ is a partition of y. We may write $\mathtt{Part}(x_i \mid i \in \{1, \dots, n\}\,; y)$ instead of $\mathtt{Part}(x_1, \dots, x_n, y)$.

◀

4.2 Definition Given $\mathbf{A} \in \Gamma^a(\mathcal{W})$, say $\mathbf{A} \leq \prod_{u \in U} \mathbf{A}_u$ with each stalk $\mathbf{A}_u \in \mathcal{W}$, we associate an Boolean algebra $\mathbf{B}_{\mathbf{A}} \overset{\mathrm{def}}{=} \langle B, \cap, \cup, \dots \rangle \in \mathcal{B}$ such that

$$\begin{aligned} B &= \{\text{all clopen subsets of the Boolean space } U\}, \\ \cap, \cup, - &= \text{set theoretic intersection, union, complement resp.,} \\ 0, 1 &= \varnothing,\ U \text{ resp.,} \\ C_m &= \{u \in U \mid |A_u| = m\} \text{ for each } m > 1. \end{aligned}$$

For each $M \geq 2$, we associate a Boolean algebra $\mathbf{B}_{\mathbf{A},M} \in \mathcal{B}_M$ which is identical with $\mathbf{B}_{\mathbf{A}}$ except that $C_m = \varnothing$ for every $m > M$. ◀

In order for this definition to make sense, $\mathtt{C}_m$'s should be clopen subsets of U, which are shown in lemma 4.3.(b) below.

4.3 Lemma *Let $\mathbf{A} \in \Gamma^a(\mathcal{W})$ and $\mathbf{B} = \mathbf{B}_{\mathbf{A}}$, and let U be the Boolean space associated with $\mathbf{A}$. For each $m > 1$, let $C_m = \mathtt{C}^{\mathbf{B}}_m$ and $C_{>m} = \mathtt{C}^{\mathbf{B}}_{>m}$. Let $\bar{a} \in A^n$ and $n \in \omega$ be given. Then,*

(a) *Every clopen subset of U is equal to some $[\![a_1 = a_2]\!]$ with $a_1, a_2 \in A$.*

(b) *C_m and $C_{>m}$ are clopen. In general, if an L-formula $\varphi(\bar{x})$ is quantifier free, then $[\![\varphi(\bar{a})]\!]$ is a member of B (i.e., a clopen subset of U) for all $\bar{a} \in A^{|\bar{x}|}$.*

(c) *$\{C_i \mid 1 < i \leq m\} \cup \{C_{>m}\}$ is a partition of U.*

(d) *C_m is partitioned by $\{[\![\theta(\bar{a})]\!] \cap C_m \mid \theta \in \Theta_m(\bar{x})\}$, and for each $k \geq 0$ and $M \geq (n+2)\cdot(k+1)$, $C_{>M}$ is partitioned by $\{[\![\theta(\bar{a})]\!] \cap C_{>M} \mid \theta \in \Theta^k_\infty(\bar{x})\}$.*

Proof. To prove (a), let X be a clopen subset of U. Let $\hat{0}, \hat{1} \in A$ be defined by $\hat{0}(u) = \mathtt{0}^{\mathbf{A}_u}$ and $\hat{1}(u) = \mathtt{1}^{\mathbf{A}_u}$ for all $u \in U$. Take $a_1 = \hat{0}$ and $a_2 = \hat{0}|_X \cup \hat{1}|_{U-X}$. Then clearly $X = [\![a_1 = a_2]\!]$.

To prove (b), first note that $|A_u| = m$ iff $\mathbf{A}_u \models \mathtt{Cycle}_m$ for all $u \in U$, $m > 1$. Since $\mathtt{Cycle}_m$ is a quantifier free sentence it suffices to show that $[\![\varphi(\bar{a})]\!]$ is clopen for all atomic formula φ. But, for atomic φ's, we have $[\![\varphi(\bar{a})]\!] = [\![c_1 = c_2]\!]$ for some $c_1, c_2 \in A$, and $[\![c_1 = c_2]\!]$ is clopen by the definition of Boolean products.

(c) is obvious.

(d) is obvious from part (c), (d) of lemma 2.5 and 2.12. □

Let us do some preparation for our Feferman-Vaught type theorem. Let $\varphi(x_1, \dots, x_n)$ be an atomic L^1-formula and let $m > 1$. Define an L_B-term t_φ and an L_B-formula φ^* by

$$\begin{aligned} t_\varphi &= \bigcup \{Y_\theta \mid \theta \in \Theta_m(\varphi)\}, \\ \varphi^* &= t_\varphi \approx \mathtt{1}. \end{aligned}$$

Let $\mathbf{A} \in \Gamma^a(\mathcal{K}_m^0)$ and $\bar{a} \in A^n$ be given. Suppose that $\mathbf{A} \models \varphi(\bar{a})$. Then, since φ is an equation, it holds in each stalk of $\mathbf{A}$: that is, $\mathbf{A}_u \models \varphi(a_1(u), \dots, a_n(u))$ for every $u \in U$, where U is the Boolean space associated with the Boolean product $\mathbf{A}$. For each $u \in U$, since $\mathbf{A}_u \in \mathcal{K}_m^0$, lemma 2.7 tells us that $\mathbf{A}_u \models \theta(\bar{a})$ for some $\theta \in \Theta_m(\varphi)$. This means $\bigcup\{[\![\theta(\bar{a})]\!] \mid \theta \in \Theta_m(\varphi)\} = U$, or in other words, $\mathbf{B_A} \models \varphi^*([\![\theta(\bar{a})]\!] \mid \theta \in \Theta_m(\varphi))$. We have shown that $\mathbf{A} \models \varphi(\bar{a}) \Rightarrow \mathbf{B_A} \models \varphi^*([\![\theta(\bar{a})]\!] \mid \theta \in \Theta_m(\varphi))$. This whole argument is reversible and hence we have

$$\mathbf{A} \models \varphi(\bar{a}) \;\Leftrightarrow\; \mathbf{B_A} \models \varphi^*([\![\theta(\bar{a})]\!] \mid \theta \in \Theta_m(\varphi)). \tag{35}$$

We want to extend above argument so that (35) applies to every L-formula φ and every $\mathbf{A} \in \Gamma^a(\mathcal{W})$. For this end we need a syntactic machinery called *reduction*, denoted $\overset{\text{red}}{\mapsto}$, for the language L over $\Gamma^a(\mathcal{W})$: that is, if φ and φ' are L-formulas and $\varphi \overset{\text{red}}{\mapsto} \varphi'$, then $\Gamma^a(\mathcal{W}) \models \varphi \leftrightarrow \varphi'$, and if t and t' are L-terms such that $t \overset{\text{red}}{\mapsto} t'$, then $\Gamma^a(\mathcal{W}) \models t \approx t'$. The definition goes as follows.

4.4 Definition For every natural number n, m and for every L-term t, s, t_1, t_2 and t_3,

$$\begin{aligned}
\mathtt{f}^n(\mathtt{g}^m(t)) &\overset{\text{red}}{\mapsto} \mathtt{f}^{n-m}(t) \quad \text{if } n \geq m,\\
&\overset{\text{red}}{\mapsto} \mathtt{g}^{m-n}(t) \quad \text{if } n < m,\\
\mathtt{g}^n(\mathtt{f}^m(t)) &\overset{\text{red}}{\mapsto} \mathtt{g}^{n-m}(t) \quad \text{if } n \geq m,\\
&\overset{\text{red}}{\mapsto} \mathtt{f}^{m-n}(t) \quad \text{if } n < m,\\
\mathtt{f}^n(t) \approx \mathtt{f}^m(s) &\overset{\text{red}}{\mapsto} \mathtt{f}^{n-m}(t) \approx s \quad \text{if } n \geq m,\\
&\overset{\text{red}}{\mapsto} \mathtt{f}^{m-n}(s) \approx t \quad \text{if } n < m,\\
\mathtt{f}^n(t) \approx \mathtt{g}^m(s) &\overset{\text{red}}{\mapsto} \mathtt{f}^{n+m}(t) \approx s,\\
\mathtt{g}^n(t) \approx \mathtt{f}^m(s) &\overset{\text{red}}{\mapsto} \mathtt{f}^{n+m}(s) \approx t,\\
\mathtt{g}^n(t) \approx \mathtt{g}^m(s) &\overset{\text{red}}{\mapsto} \mathtt{f}^m(t) \approx \mathtt{f}^n(s),\\
\mathtt{f}^n(\mathtt{t}(t_1, t_2, t_3)) &\overset{\text{red}}{\mapsto} \mathtt{t}(\mathtt{f}^n(t_1), \mathtt{f}^n(t_2), \mathtt{f}^n(t_3)),\\
\mathtt{g}^n(\mathtt{t}(t_1, t_2, t_3)) &\overset{\text{red}}{\mapsto} \mathtt{t}(\mathtt{g}^n(t_1), \mathtt{g}^n(t_2), \mathtt{g}^n(t_3)).
\end{aligned}$$

◀

It is clear that we can set up some definite recursive reduction scheme by which we obtain a unique formula φ' from a given L-formula φ so that φ' cannot be reduced further. Now we are ready to prove our Feferman-Vaught type theorem.

4.5 Theorem *Given any L-formula $\varphi(x_1, \dots, x_n)$ we can effectively find an integer $k(\varphi) \geq 0$ with the following property: for every $k \geq k(\varphi)$, we can effectively an L_B-formula $\varphi^*(Y_\delta \mid \delta \in \Delta(n, k))$, depending on k, such that for every $\mathbf{A} \in \Gamma^a(\mathcal{W})$ and for every $\bar{a} \in A^n$, we have the following biimplication:*

$$\mathbf{A} \models \varphi(\bar{a}) \;\Leftrightarrow\; \mathbf{B_A} \models \varphi^*([\![\delta(\bar{a})]\!] \mid \delta \in \Delta(n, k)). \tag{36}$$

Moreover φ^* can be chosen so that

$$\text{For each } m > (n+2)\cdot(k+1), \text{ the symbol } \mathtt{C}_m \text{ does not occur in } \varphi^*. \tag{37}$$

PROOF. Throughout this proof, $\mathbf{A} \leq \prod_{u \in U} \mathbf{A}_u$ is a Boolean product with each stalk $\mathbf{A}_u \in \mathcal{W}$.

The proof proceeds by induction on the complexity of φ. The condition (37) is easily checked on each induction stage, and we are not going to mention this henceforth in this proof.

Suppose that φ is atomic. We will find an integer $k(\varphi) \geq 0$ and construct an L_B-term $t_\varphi(Y_\delta \mid \delta \in \Delta)$, depending on $k \geq k(\varphi)$, so that for all $\mathbf{A} \in \Gamma^a(\mathcal{W})$,

$$t_\varphi([\![\delta(\bar{a})]\!] \mid \delta \in \Delta) = \{u \in U \mid \mathbf{A}_u \models \varphi(a_1(u), \ldots, a_n(u))\} \tag{38}$$

by induction on the number $N_{\mathtt{t}}(\varphi)$ of occurrences of the discriminator function symbol $\mathtt{t}$ in φ. Then by letting

$$\varphi^* = t_\varphi \approx \mathtt{1}$$

we will be done.

Let $k(\varphi)$ be the number of all occurrences of the symbols $\mathtt{f}$ and $\mathtt{g}$ in φ. Given $k \geq k(\varphi)$, let $M = (n+2)\cdot(k+1)$.

If $N_{\mathtt{t}}(\varphi) = 0$, then by the reduction scheme in definition 4.4, φ may be assumed to be of the form $\mathtt{f}^q(x) \approx y$ where $q \leq k$ and $\{x, y\} \subseteq \{x_1, \ldots, x_n, \mathtt{0}, \mathtt{1}\}$. So if we let

$$t_\varphi = \bigcup \{Y_\delta \cap \mathtt{C}_m \mid \delta \in \Theta_m(\varphi),\ m = 2, \ldots, M\} \cup \bigcup \{Y_\delta \cap \mathtt{C}_{>M} \mid \delta \in \Theta^k_\infty(\varphi)\},$$

then by lemma 4.3.(c) and 4.3.(d), lemma 2.7 and lemma 2.15, the desired equality (38) and the biimplication (36) should hold.

If $N_{\mathtt{t}}(\varphi) > 0$, then let $\varphi = \mathtt{t}(s^0, s^1, s^2) \approx s^3$ for some L-terms s^i's with $i < 4$. (There is no loss of generality here because of the reduction scheme 4.4.) We let

$$t_\varphi = (t_{s^0 \approx s^1} \cap t_{s^2 \approx s^3}) \cup (-t_{s^0 \approx s^1} \cap t_{s^0 \approx s^3}).$$

Then the following series of biimplications holds. (We use the notation $\bar{a}(u)$ as an abbreviation for $\langle a_1(u), \ldots, a_n(u)\rangle$.)

$$\begin{aligned}
\mathbf{A} \models \varphi(\bar{a}) &\Leftrightarrow \mathbf{A} \models \mathtt{t}(s^0, s^1, s^2)(\bar{a}) \approx s^3(\bar{a}) \\
&\Leftrightarrow (\forall u \in U)\Big(\big[s^0(\bar{a}(u)) = s^1(\bar{a}(u)) \wedge s^3(\bar{a}(u)) = s^2(\bar{a}(u))\big] \vee \\
&\qquad\qquad \big[s^0(\bar{a}(u)) \neq s^1(\bar{a}(u)) \wedge s^3(\bar{a}(u)) = s^0(\bar{a}(u))\big]\Big) \\
&\Leftrightarrow \big[\{u \in U \mid s^0(\bar{a}(u)) = s^1(\bar{a}(u))\} \cap \{u \in U \mid s^3(\bar{a}(u)) = s^2(\bar{a}(u))\}\big] \cup \\
&\qquad \big[\{u \in U \mid s^0(\bar{a}(u)) \neq s^1(\bar{a}(u))\} \cap \{u \in U \mid s^3(\bar{a}(u)) = s^0(\bar{a}(u))\}\big] \\
&\quad = U
\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \big[t_{s^0\approx s^1}(\llbracket\delta(\bar a)\rrbracket \bigm| \delta\in\Delta)\ \cap\ t_{s^3\approx s^2}(\llbracket\delta(\bar a)\rrbracket \bigm| \delta\in\Delta)\big]\cup\\
&\qquad \big[t_{s^0\not\approx s^1}(\llbracket\delta(\bar a)\rrbracket \bigm| \delta\in\Delta)\ \cap\ t_{s^3\approx s^0}(\llbracket\delta(\bar a)\rrbracket \bigm| \delta\in\Delta)\big]\\
&\quad = U \quad \text{(by the induction hypothesis (38))}\\
&\Leftrightarrow t_\varphi(\delta(\bar a) \bigm| \delta\in\Delta) = U\\
&\Leftrightarrow \mathbf{B_A} \models \varphi^*(\llbracket\delta(\bar a)\rrbracket \bigm| \delta\in\Delta)
\end{aligned}$$

This completes the proof for the case when φ is atomic.

Next suppose that $\varphi(\bar x) = \neg\psi(\bar x)$ for some ψ. Then we just take $k(\varphi) = k(\psi)$ and $\varphi^* = \neg\psi^*$ (for each $k \geq k(\varphi)$). If $\varphi(\bar x) = \psi_1(\bar x) \wedge \psi_2(\bar x)$ for some ψ_1 and ψ_2, then let $k(\varphi) = \max(k(\psi_1), k(\psi_2))$ and let $\varphi^* = \psi_1^* \wedge \psi_2^*$.

Finally, suppose that $\varphi(\bar x) = \exists y\psi(\bar x, y)$ for some $\psi(\bar x, y)$. Then let $k(\varphi) = 2\cdot k(\psi)$, and for each $k \geq k(\varphi)$ let

$$\varphi^*\big(Y_\delta \bigm| \delta\in\Delta(n,k)\big) = \big(\exists Y_\theta \bigm| \theta\in\Delta(n+1,k(\psi))\big)\Big(\tag{39}$$

$$\psi^*\big(Y_\theta \bigm| \theta\in\Delta(n+1,k(\psi))\big)\ \wedge \tag{40}$$

$$\bigwedge_{m=2}^{M(\psi)} \mathtt{Part}\big(Y_\theta\cap \mathtt{C}_m \bigm| \theta\in\Theta_m(\bar x,y)\,;\mathtt{C}_m\big)\ \wedge \tag{41}$$

$$\mathtt{Part}(Y_\theta\cap \mathtt{C}_{>M(\psi)} \bigm| \theta\in\Theta_\infty^{k(\psi)}(\bar x,y)\,;\mathtt{C}_{>M(\psi)})\ \wedge \tag{42}$$

$$\bigwedge_{m=2}^{M(\psi)}\ \bigwedge_{\theta\in\Delta(n+1,k(\psi))} \big(Y_\theta\cap\mathtt{C}_m \subseteq \bigcup\{Y_\delta \bigm| \delta\in\Delta_{1,m}\}\big)\ \wedge \tag{43}$$

$$\bigwedge_{\theta\in\Delta(n+1,k(\psi))} \big(Y_\theta\cap\mathtt{C}_{>M(\psi)} \subseteq \bigcup\{Y_\delta \bigm| \delta\in\Delta_2\}\big)\Big), \tag{44}$$

where

$$\begin{aligned}
M(\psi) &\stackrel{\text{def}}{=} ((n+1)+2)\cdot(k(\psi)+1),\\
\Delta_{1,m} &\stackrel{\text{def}}{=} \{\delta\in\Theta_m(\bar x) \bigm| \delta\wedge\theta \text{ has a model in } \mathcal{K}_m^0\},\\
\Delta_2 &\stackrel{\text{def}}{=} \{\delta\in\Theta_\infty^k(\bar x) \bigm| \delta\wedge\theta \text{ has a model in } \mathcal{K}_\infty^0\}.
\end{aligned}$$

We will assume without loss of generality[3] that $\Delta(n,k)\cap\Delta(n+1,k(\psi)) = \varnothing$ so that the existential quantification in (39) has no effect on Y_δ's. Note that the map $\varphi\mapsto\varphi^*$ is recursive since the theories of $\mathcal{K}_m^0$, $(m>1)$ and $\mathcal{K}_\infty^0$ are all decidable as was mentioned earlier.

Given any $\mathbf{A}\in\Gamma^a(\mathcal{W})$ and $\bar a\in A^n$, we need to show that this φ^* satisfies (36).

To prove $\Rightarrow$ of (36), suppose that $\mathbf{A}\models\varphi(\bar a)$. Then $\mathbf{A}\models\psi(\bar a,b)$ for some $b\in A$. For each $\theta\in\Delta(n+1,k(\psi))$, we choose $\llbracket\theta(\bar a,b)\rrbracket$ as witness of Y_θ in (39). Then (40) clearly holds by induction hypothesis. (41) and (42) hold because of part (c),(d) of lemma 2.5 and 2.12 respectively.

[3]To be rigorous, we assign, for each ordered pair $\langle\varphi,\delta\rangle$ of L-formulas, a unique L_B-variable Y_δ^φ.

To show (43), let $m \in \{2, \ldots, M(\psi)\}$ and $\theta \in \Delta(n+1, k(\psi))$ be given. We want to show

$$[\![\theta(\bar{a}, b)]\!] \cap C_m \subseteq \bigcup \{[\![\delta(\bar{a})]\!] \mid \delta \wedge \theta \text{ has a model in } \mathcal{K}_m^0,\ \delta \in \Theta_m(\bar{x})\} \tag{45}$$

Let $u \in [\![\theta(\bar{a}, b)]\!] \cap C_m$. Then $|A_u| = m$ and $\mathbf{A}_u \models \theta(a_1(u), \ldots, a_n(u), b(u))$. We want to show that this u belongs to the r.h.s. of (45). For this end just let $\delta = \bigwedge \Psi^m_{\mathbf{A}_u, \bar{a}}$. Then clearly $\mathbf{A}_u \in \mathcal{K}_m^0$ is a model of $\delta \wedge \theta$ and $u \in [\![\delta(\bar{a})]\!]$. This proves (43). We can prove (44) similarly, using lemma 2.9 this time.

Now proving $\Leftarrow$ of (36) is perhaps more interesting. Suppose that there are clopen subsets $\{E_\theta \mid \theta \in \Delta(n+1, k(\psi))\}$ of U for which (39)–(44) hold (in $\mathbf{B_A}$) with each Y_θ replaced by E_θ and each Y_δ replaced by $[\![\delta(\bar{a})]\!]$ respectively. We want to show that $\mathbf{A} \models \varphi(\bar{a})$, or equivalently $\mathbf{A} \models \psi(\bar{a}, b)$ for some $b \in A$. By induction, it suffices to find $b \in A$ such that

$$\psi^*([\![\theta(\bar{a}, b)]\!] \mid \theta \in \Delta(n+1, k(\psi))) \tag{46}$$

holds. But we know $\psi^*(E_\theta \mid \theta \in \Delta(n+1, k(\psi)))$ by supposition. So (46) will follow from $[\![\theta(\bar{a}, b)]\!] = E_\theta$ for all θ, or equivalently, $[\![\theta(\bar{a}, b)]\!] \cap C_m = E_\theta \cap C_m$ for each $m = 2, \ldots, M(\psi)$ and $[\![\theta(\bar{a}, b)]\!] \cap C_{>M(\psi)} = E_\theta \cap C_{>M(\psi)}$, for all θ. Further, we know that each $E_\theta \cap C_m$ is partitioned by $\{[\![\delta(\bar{a})]\!] \mid \delta \in \Delta_{1,m}\}$ and each $E_\theta \cap C_{>M(\psi)}$ is partitioned by $\{[\![\delta(\bar{a})]\!] \mid \delta \in \Delta_2\}$ from (43), (44), lemma 2.5, lemma 2.12 and the definition of $\Delta_{1,m}$'s and Δ_2.

Thus what we want to do is to find $b \in A$ such that for each $\theta \in \Delta(n+1, k(\psi))$ the following hold:

(a) for each $m = 2, \ldots, M(\psi)$ and for each $\delta \in \Delta_{1,m}$,

$$[\![\theta(\bar{a}, b)]\!] \cap C_m \cap [\![\delta(\bar{a})]\!] = E_\theta \cap C_m \cap [\![\delta(\bar{a})]\!] \tag{47}$$

(b) for each $\delta \in \Delta_2$,

$$[\![\theta(\bar{a}, b)]\!] \cap C_{>M(\psi)} \cap [\![\delta(\bar{a})]\!] = E_\theta \cap C_{>M(\psi)} \cap [\![\delta(\bar{a})]\!] \tag{48}$$

Basically we construct b piecewise on each $E_\theta \cap C_m \cap [\![\delta(\bar{a})]\!]$ and $E_\theta \cap C_{>M} \cap [\![\delta(\bar{a})]\!]$ with the desired properties (47) and (48) respectively, and then use the patchwork property to join all these pieces into a map on the entire set U.

Let $E \stackrel{\text{def}}{=} E_\theta \cap C_m \cap [\![\delta(\bar{a})]\!]$ be given. First we show $\supseteq$ of (47) as follows. Assume E is nonempty without loss of generality. For each $u \in E$, since $\mathbf{A}_u \models \delta$ and $\delta \wedge \theta$ has a model in $\mathcal{K}_m^0$, this means that $\mathbf{A}_u$ is in fact a model of θ by lemma 2.8.[4] That is, there must exist $b_u \in A_u$ such that $\mathbf{A}_u \models \theta(a_1(u), \ldots, a_n(u), b_u)$.

Since $\mathbf{A}$ is a subdirect product, there exists $b^u \in A$ such that $b^u(u) = b_u$. Then clearly $\{[\![\theta(\bar{a}, b^u)]\!] \cap \mathtt{C}_m \cap [\![\delta(\bar{a})]\!] \mid u \in E\}$ is an open cover of E, which is compact since it is a

[4] $\exists y \theta(\bar{x}, y)$ and δ play the roles of φ and θ in lemma 2.8 respectively.

closed subset of the Boolean space U. Thus there exist $u_1, \dots, u_s \in E$ for some $s > 0$ such that $\{[\![\theta(\bar{a}, b^{u_i})]\!] \cap \mathtt{C}_m \cap [\![\delta(\bar{a})]\!] \mid i = 1, \dots, s\}$ covers E. Using an elementary trick we can assume without loss of generality that $\{[\![\theta(\bar{a}, b^{u_i})]\!] \cap \mathtt{C}_m \cap [\![\delta(\bar{a})]\!] \mid i = 1, \dots, s\} \stackrel{\text{def}}{=} \{E_i \mid i = 1, \dots, s\}$ forms a partition of E. Now let $b_E = \bigcup_{i=1}^{s} b^{u_i}|_{E_i}$ be a map on E. We have shown that, if we replace b by b_E in (47), then $\supseteq$ holds for this particular m and δ.

The other direction $\subseteq$ of (47) is obvious from the construction: i.e., $[\![\theta(\bar{a}, b_E)]\!] \subseteq E$ holds from the construction of the map b_E. Then clearly the (disjoint) union b of all such b_E's will have the property (47) for all m, δ and θ.

It remains to handle E's of the form $E_\theta \cap C_{>M(\psi)} \cap [\![\delta(\bar{a})]\!]$, which can easily be done similarly using lemma 2.11. $\square$

5 The Main Result

In the following discussions recall that the constant symbols $\mathtt{0}$ and $\mathtt{1}$ are used in both L and L_B, where L is the language of our variety $\mathcal{V}$ and L_B is the language of the class $\mathcal{B}$ of all Boolean algebras with denumerably many constants, which is known to be decidable.

Given a member $\mathbf{A}$ of $\mathcal{V}$, there exists a Boolean product $\mathbf{A}'$ with stalks from $\mathcal{W}$ such that $\mathbf{A}'$ is isomorphic to $\mathbf{A}$. In a stalk of $\mathbf{A}'$, $\mathtt{0}$ and $\mathtt{1}$ are interpreted as two distinguished members 0 and 1 such that $0 + n = 1 \Rightarrow 1 + h(n) = 0$ where h is our fixed recursive function with a nonrecursive range.

In a Boolean algebra, $\mathtt{0}$ is interpreted as the least element and $\mathtt{1}$ is interpreted as the largest element.

We first state and prove an easy but essential lemma.

5.1 Lemma *Let σ be an L_B sentence such that no constant symbol $\mathtt{C}_m$ with $m > M$ occurs in it. Then σ has a model in $\mathcal{B}$ if and only if σ has a model in $\mathcal{B}_M$.*

Proof. If $\mathbf{B}_M \in \mathcal{B}_M$ is obtained from $\mathbf{B} \in \mathcal{B}$ by reinterpreting all constant symbols $\mathtt{C}_m$ with $m > M$ as the emptyset, then $\mathbf{B}_M \models \sigma \Leftrightarrow \mathbf{B} \models \sigma$. $\square$

5.2 Theorem *The variety $\mathcal{V}$ defined in definition 3.1 is decidable but not finitely decidable.*

Proof. The finite undecidability of $\mathcal{V}$ has already been proved in theorem 3.2.

To prove the decidability of $\mathcal{V} = \mathsf{I}\Gamma^a(\mathcal{W})$, we will construct a recursive map $\varphi \mapsto \varphi^\#$ from the set of L-sentences into the set of L_B-sentences so that

$$\varphi \text{ has a model in } \mathcal{V} \Leftrightarrow \varphi^\# \text{ has a model in } \mathcal{B}, \tag{49}$$

where $\mathcal{B}$ is the class of all Boolean algebras with denumerably many constants $\mathtt{C}_m$ $(m \geq 2)$. Let an L-sentence φ be given. First let $k = k(\varphi)$ and obtain an L_B-sentence $\varphi^*\big(Y_\delta \mid Y_\delta \in$

$\Delta(0,k)\big)$ as in (36). Then let

$$\varphi^{\#} = (\exists Y_\delta \mid \delta \in \Delta(0,k))\Big(\varphi^*(Y_\delta \mid \delta \in \Delta(0,k)) \wedge \tag{50}$$

$$\bigwedge_{m=2}^{M} \mathtt{Part}(Y_\delta \cap \mathtt{C}_m \mid \delta \in \Delta_{1,m}; \mathtt{C}_m) \wedge \tag{51}$$

$$\mathtt{Part}(Y_\delta \cap \mathtt{C}_{>M} \mid \delta \in \Delta_2; \mathtt{C}_{>M}) \wedge \tag{52}$$

$$\mathtt{Part}(\mathtt{C}_2, \dots, \mathtt{C}_M, \mathtt{C}_{>M}, \mathtt{1})\Big), \tag{53}$$

where

$$\begin{aligned}
\varphi^* &\text{ is obtained as in (36),}\\
M &\stackrel{\text{def}}{=} (0+2)\cdot(k+1),\\
\Delta_{1,m} &\stackrel{\text{def}}{=} \{\delta \in \Theta_m(\langle\,\rangle) \mid \delta \text{ has a model in } \mathcal{K}^1_m\},\\
\Delta_2 &\stackrel{\text{def}}{=} \{\delta \in \Theta^k_\infty(\langle\,\rangle) \mid \delta \text{ has a model in } \mathcal{K}^1_\infty\}.
\end{aligned}$$

Note that the map $\varphi \mapsto \varphi^{\#}$ is recursive since the theories of $\mathcal{K}^1_m$, $(m > 1)$ and $\mathcal{K}^1_\infty$ are all decidable by corollary 2.18. Also note that no constant symbol $\mathtt{C}_m$ with $m > M$ occurs in $\varphi^{\#}$.

Using lemma 5.1, the proof consists of the following two parts:

(a) If φ has a model $\mathbf{A} \in \Gamma^a(\mathcal{W})$, then $\mathbf{B}_{\mathbf{A},M} \in \mathcal{B}_M$ models $\varphi^{\#}$.

(b) If $\mathbf{B} \in \mathcal{B}_M$ models $\varphi^{\#}$, then there exists $\mathbf{A} \in \Gamma^a(\mathcal{W})$ such that $\mathbf{B}_{\mathbf{A},M}$ is isomorphic to $\mathbf{B}$ and $\mathbf{B}_{\mathbf{A},M} \models \varphi^*(\llbracket\delta\rrbracket \mid \delta \in \Delta(0,k))$, which implies $\mathbf{A} \models \varphi$ because of (36) and (50).

To prove (a), the easier part, let φ be an L-sentence, and let a Boolean product $\mathbf{A} \le \prod_{u\in U} \mathbf{A}_u$, a member of $\Gamma^a(\mathcal{W})$, be given so that $\mathbf{A} \models \varphi$. We want to show that $\mathbf{B}_{\mathbf{A},M} \models \varphi^{\#}$. First, choose $\langle \llbracket\delta\rrbracket \mid \delta \in \Delta(0,k)\rangle$ as a witness for $\langle Y_\delta \mid \delta \in \Delta(0,k)\rangle$ in $\varphi^{\#}$. Then (50) holds by our Feferman-Vaught type theorem 4.5.

To show (51), let $m \in \{2, \dots, M\}$. Then for each $u \in C_m$, we have $\mathbf{A}_u \models \delta$ for some $\delta \in \Delta_{1,m}$ because of lemma 2.5.(c) and our definition of $\Delta_{1,m}$. This shows $\bigcup_{\delta\in\Delta_{1,m}} \llbracket\delta\rrbracket \cap C_m = C_m$. Now (51) follows from this and lemma 2.5.(d).

(52) is similarly proved using lemma 2.12. (53) follows from lemma 4.3.(c). This completes the proof of part (a).

Now to prove part (b), let φ be an L-sentence and suppose that $\mathbf{B} \in \mathcal{B}_M$ models $\varphi^{\#}$. We are going to construct $\mathbf{A} \in \Gamma^a(\mathcal{W})$ such that $\mathbf{B}_{\mathbf{A},M} \cong \mathbf{B}$ and $\mathbf{B}_{\mathbf{A},M} \models \varphi^*(\llbracket\delta\rrbracket \mid \delta \in \Delta(0,k))$.

By the Stone duality, we may assume that the universe B of $\mathbf{B}$ is the collection of all clopen subsets of a Boolean topology U, and $\cap^{\mathbf{B}}, \cup^{\mathbf{B}}, -^{\mathbf{B}}$ are respectively set theoretic intersection, union, complement on U, and $\mathtt{0}^{\mathbf{B}}$ and $\mathtt{1}^{\mathbf{B}}$ are respectively the emptyset and the whole set U, and finally $\mathtt{C}^{\mathbf{B}}_m$'s are clopen subsets of U such that $\mathtt{C}^{\mathbf{B}}_m = \varnothing$ for all $m > M$.

As $\mathbf{B} \models \varphi^{\#}$, we can take a sequence $\langle E_\delta \mid \delta \in \Delta(0,k)\rangle$ of clopen subsets of U as a witness for $\langle Y_\delta \mid \delta \in \Delta(0,k)\rangle$ in (50)–(53). We first construct a family $\{\mathbf{A}_u\}_{u\in U}$ of L-structures, and then define a subdirect subuniverse $A \in \prod_{u\in U} A_u$ with the desired properties. We use the fact that U is partitioned by $\{C_2,\dots,C_M,C_{>M}\}$ by (53).

If $u \in C_m$ for some $m \in \{2,\dots,M\}$, then first choose, using (51), a formula $\delta \in \Delta_{1,m}$ so that $u \in E_\delta$, and then choose $\mathbf{A}_u \in (\mathcal{K}^1_m)^{\mathtt{t}}$ so that $\mathbf{A}_u \models \delta$. Without loss of generality we assume that $A_u = \mathbb{Z}_m \stackrel{\text{def}}{=} \{0,\dots,m-1\}$ and $\mathtt{0}^{\mathbf{A}_u} = 0$.

If $u \in C_{>M}$, then first choose, using (52), a formula $\delta \in \Delta_2$ so that $u \in E_\delta$, and then choose $\mathbf{A}_u \in (\mathcal{K}^1_{>M})^{\mathtt{t}}$ so that $\mathbf{A}_u \models \delta$. Since $\mathbf{B}$ was chosen from $\mathcal{B}_M$, $|A_u| > M$ means A_u is an infinite set. Thus without loss of generality we can assume that $A_u = \mathbb{Z}$ and $\mathtt{0}^{\mathbf{A}_u} = 0$ by lemma 2.9.

It remains to define $A \subseteq \prod_{u\in U} A_u$ so that $\mathbf{A}$ is a Boolean product, $\mathbf{B}_{\mathbf{A},M} \cong \mathbf{B}$ and $\mathbf{B}_{\mathbf{A},M} \models \varphi^*(\llbracket\delta\rrbracket \mid \delta \in \Delta(0,k))$: i.e., we want the following:

(i) A is closed under the basic operations.

(ii) $A \subseteq \prod_{u\in U} A_u$ is subdirect: i.e., $(\forall u \in U)(\forall y \in A_u)(\exists a \in A)\big(a(u) = y\big)$.

(iii) $\llbracket a = b\rrbracket$ is a clopen subset of U for all $a, b \in A$.

(iv) A has the patchwork property.

(v) For each $m \geq 2$, we have $\mathtt{C}_m^{\mathbf{B}_{\mathbf{A},M}} = \mathtt{C}_m^{\mathbf{B}}$.

(vi) For every $\delta \in \Delta(0,k)$, we have $E_\delta = \llbracket\delta\rrbracket$.

For this end define a map $\hat{\imath}_Y : Y \to \mathbb{Z}$ for each $\langle i, Y\rangle \in \mathbb{Z} \times B$ satisfying

$$i \in A_u \text{ for all } u \in Y \tag{54}$$

by

$$\hat{\imath}_Y = \{\langle u, i\rangle \mid u \in Y\}.$$

With these collections of $\hat{\imath}_Y$'s, which are sort of "constant maps on Y with value i", we define

$$A = \{\hat{\imath}_{0\,Y_0} \cup \dots \cup \hat{\imath}_{k-1\,Y_{k-1}} \mid k \geq 1,\ \{Y_0,\cdots,Y_{k-1}\} \text{ partitions } U,$$
$$\langle i_j, Y_j\rangle \in \mathbb{Z} \times B \text{ satisfies (54) for each } j < k\}.$$

Intuitively, A is the collection of all maps on U which are patchworks of finitely many constant maps. Let us prove (i)–(vi) in the following order: (v), (ii), (iii), (iv), (vi), (i).

(v) is obvious from the construction: that is, $\mathtt{C}_m^{\mathbf{B}_{\mathbf{A},M}} = \{u \in U \mid |A_u| = m\} = \mathtt{C}_m^{\mathbf{B}}$ for $m = 2,\dots,M$ and for $m > M$.

(ii), (iii) and (iv) are straightforwardly checked.

To check (vi), by (53), it suffices to show, for all $\delta \in \Delta(0,k)$,

$$E_\delta \cap C_m = [\![\delta]\!] \cap C_m \text{ for each } m = 2, \dots, M, \text{ and} \tag{55}$$

$$E_\delta \cap C_{>M} = [\![\delta]\!] \cap C_{>M}. \tag{56}$$

Moreover for (55), we only need to check this equality for $\delta \in \Delta_{1,m}$ because of (51) and the construction of $\mathbf{A}_u$'s. Similarly only $\delta \in \Delta_2$ need be checked for (56). In both cases, $\subseteq$ directly follows from the construction of $\mathbf{A}_u$'s. For $\supseteq$ of (55), just recall that for $|A_u| = m$, $\delta \in \Delta_{1,m}$, $\delta = \delta'$ is implied by ($\mathbf{A}_u \models \delta$ and $\mathbf{A}_u \models \delta'$). $\supseteq$ of (56) is similarly proved.

It remains to show (i). To show that A is closed under f, let $a \in A$ be given and let i_Y be a constituent constant map of a. It suffices to show that $f(i_Y)$ is a patchwork of two constant maps. Define two subsets Y_1, Y_2 of Y so that $Y_1 \stackrel{\text{def}}{=} Y \cap C_{i+1}$ and $Y_2 \stackrel{\text{def}}{=} Y \cap C_{>i+1}$. Then it is clear that Y_1, Y_2 partition Y and they are both clopen. Thus $f(\hat{\imath}_Y) = \hat{0}_{Y_1} \cup (\widehat{i+1})_{Y_2}$.

To show that A is closed under g, let $a \in A$ be given and let $\hat{\imath}_Y$ be a constituent constant map of a. $g(\hat{\imath}_Y)$ will be simply equal to $(\widehat{i-1})_Y$ unless $i = 0$. So let us assume that $i = 0$. In this case, we partition Y into M mutually disjoint clopen subsets as follows: $Y_j = Y \cap C_{j+1}$ for $j = 1, \dots, M-1$ and $Y_M = Y \cap C_{>M}$. Then it is easy to verify that $g(\hat{0}_Y) = \hat{1}_{Y_1} \cup \cdots (\widehat{M-1})_{Y_{M-1}} \cup (\widehat{-1})_{Y_M}$.

Finally, for the discriminator operation t, just note that $t(a,b,c) = a_{Y_1} \cup c_{Y_2}$ for all $a, b, c \in A$, where $Y_1 \stackrel{\text{def}}{=} [\![a \neq b]\!]$ and $Y_2 \stackrel{\text{def}}{=} [\![a = b]\!]$. $\square$

6 Discussions and Acknowledgements

Let T be the first-order theory of a variety $\mathcal{U}$. Then the implication

$$T \text{ decidable} \Rightarrow T_{\text{fin}} \text{ decidable.} \tag{57}$$

holds for the following situations:

(a) $\mathcal{U}$ is a variety of groups. [Zam78]

(b) $\mathcal{U}$ is a variety of rings. [Zam76]

(c) $\mathcal{U}$ is locally finite and strongly abelian. [MV89]

(d) $\mathcal{U}$ is finitely generated and congruence distributive. [Idz89]

(e) $\mathcal{U}$ is locally finite discriminator with unary stalks. [Wil93]

Among these five situations (e) is particularly interesting in that it fits so closely to our variety $\mathcal{V}$ (for which (57) fails). The only condition in (e) that $\mathcal{V}$ fails to satisfy is the local finiteness. We would like to ask the following question:

Problem: When does a theory T satisfy (57)?

More specifically,

(a) Does the theory of a finitely generated (or locally finite) variety always satisfy (57)?

(b) Does every finitely axiomatizable first-order theory satisfy (57)?

The original version of this paper is part of the author's Ph.D. thesis [Jeo91]. In this revised version, the construction of our variety $\mathcal{V}$ is a little simplified and the proof is much streamlined. The author wishes to express his deep thanks to his advisor R. McKenzie for his contribution to this paper.

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